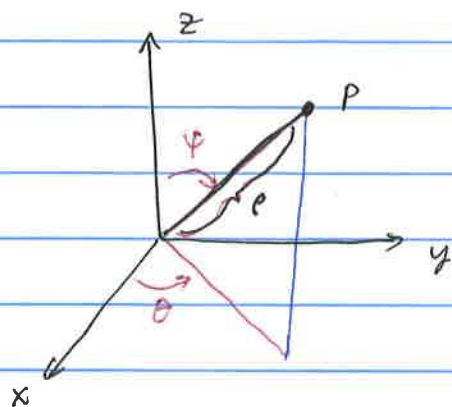


§ 12.7 : Triple Integrals in Spherical Coords



any point $P \in \mathbb{R}^3$ lies on a sphere centered at the origin of radius $\rho = \sqrt{x^2 + y^2 + z^2}$.

Cut the sphere with a plane Π through $\vec{0}$ and $\vec{P}$, $\perp$ to xy -plane, this gives the angle θ .

Then φ is the angle in the plane Π that gives P .

To make coordinates unique we demand that

$$\rho \geq 0$$

$$0 \leq \varphi \leq \pi$$

$$0 \leq \theta < 2\pi$$

$\rho = c$: sphere w/ radius c .

$\theta = c$: a half-plane (since $\rho \geq 0$)

$0 < \theta < \pi/2$, $\varphi = c$ a half-cone (upper)

$\pi/2 < \theta < \pi$ (lower half)

Conversions.

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

$$z = \rho \cos \varphi$$

$$r = \rho \sin \varphi$$

} Convert to polar, then to spherical.

or cylindrical

Ex. Convert $(0, 2\sqrt{3}, -2)$ to spherical coords.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = \sqrt{16} = 4$$

so then $0 = 4 \sin \varphi \cos \theta$

$$2\sqrt{3} = 4 \sin \varphi$$

$$-2 = 4 \cos \varphi \Rightarrow \cos \varphi = -1/2 \Rightarrow \varphi = 2\pi/3$$

then $\cos \theta = 0, \sin \varphi \neq 0 \Rightarrow \theta = \pi/2$.

and $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$.

Thus $(0, 2\sqrt{3}, -2)$ is $(4, 2\pi/3, \pi/2)$ in spherical (ρ, θ, φ) coords.

The volume element in spherical coords:

$$dV = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi \quad (*) \quad \text{Proj 4?}$$

Then, if E is a spherical wedge (or spherical rectangle) then

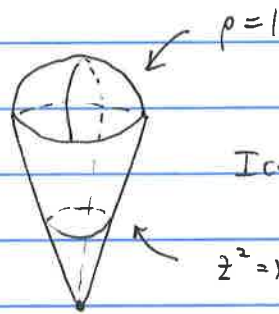
$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho, \theta, \varphi) \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi.$$

Ex. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$ w/ $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$.

$$f(\rho, \theta, \varphi) = e^{\rho^3} \quad \rho: 0 \rightarrow 1, \quad \theta: 0 \rightarrow 2\pi, \quad \varphi: 0 \rightarrow \pi$$

$$\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \sin \varphi \, d\rho \, d\theta \, d\varphi = \dots = \frac{4}{3}(\pi e - \pi)$$

Ex. (31) Find the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.



Ice cream cone shape

$$z^2 = x^2 + y^2$$

$$z^2 = \rho^2$$

$$\rho^2 \cos^2 \varphi = \rho^2 \sin^2 \varphi$$

$$1 = \frac{\sin^2 \varphi}{\cos^2 \varphi}$$

$$\tan \varphi = \pm 1$$

$$\varphi = \pm \pi/4, 3\pi/4$$

$$\text{bc } \varphi \in [0, \pi].$$

So, integral becomes

$$\int_{-\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{1}{3} \int_{-\pi/4}^{3\pi/4} \int_0^{2\pi} \sin \varphi \, d\theta \, d\varphi$$

$$= \frac{2\pi}{3} \int_{-\pi/4}^{3\pi/4} \sin \varphi \, d\varphi$$

$$= -\frac{2\pi}{3} \cdot \cos \varphi \Big|_{-\pi/4}^{3\pi/4} = -\frac{2\pi}{3} \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = \boxed{\frac{2\sqrt{2}\pi}{3}}$$