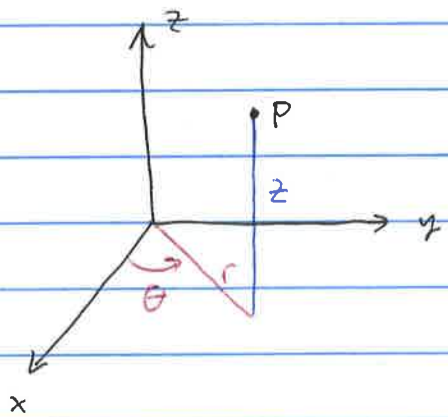
Section 12.6: Triple Integrals in Cylindrical Coordinates

Recall: In our last example of §12.5 we evaluated the z-integral, then found it would be easier to use polar coords to evaluate the xy-integral. We actually used <u>Cylindrical Coordinates</u>:



$P = (r, \theta, z)$

$x = r\cos\theta$

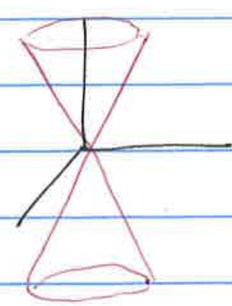
$y = r\sin\theta$

<u>Ex.</u> Convert $(2, 2\pi/3, 1)$ to rectangular coords.

$$\left.\begin{aligned} x &= 2\cos\left(\tfrac{2\pi}{3}\right) = 2\left(-\tfrac{1}{2}\right) = -1 \\ y &= 2\sin\left(\tfrac{2\pi}{3}\right) = 2\left(\tfrac{\sqrt{3}}{2}\right) = \sqrt{3} \\ z &= 1 \end{aligned}\right\} \quad (-1, \sqrt{3}, 1).$$

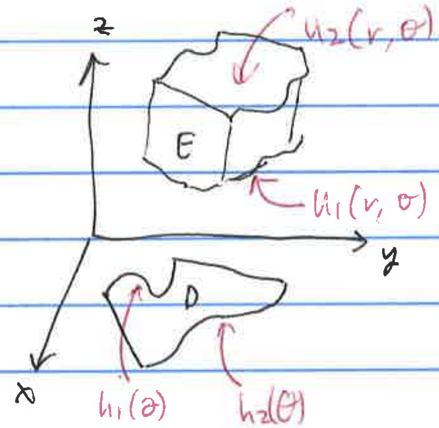
<u>Ex.</u> Describe the surface $z = r$.

It's a cone!



Evaluating triple integrals in cylindrical. A usual argument (make a rectangle enclosing the domain, then "shrink it") yields:

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$



<u>Ex</u>. A solid lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$.

Find the volume of the solid

$$V(E) = \iiint_E 1\,dV.$$

$\theta=0 \quad r=0 \quad z=1-r^2$

$\theta=2\pi \quad r=1 \quad z=4$

$$V(E) = \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} 1\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r-(r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r+r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{3}{2}r^2+\tfrac{1}{4}r^4\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} 7/4\,d\theta$$

$$= 7/4(2\pi) = \boxed{\tfrac{7}{2}\pi}$$

Ex. Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$

$f(r, \theta, z) = r^2$

$z: r \to 2$

$y: -\sqrt{4-x^2} \to \sqrt{4-x^2}$

i.e., $r: 0 \to 2$

$\theta: 0 \to 2\pi$

so the integral becomes:

$$\int_0^{2\pi} \int_0^2 \int_r^2 r^2 \, r \, dz\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^3(2-r)\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 2r^3 - r^4\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}r^4 - \frac{1}{5}r^5 \Big|_0^2 \, d\theta$$
$$= 2\pi\left(8 - 32/5\right) = \boxed{16\pi/5}$$