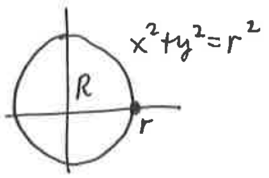
§ 12.3: Double Integrals in Polar coords

Recall: Polar coordinates:



$r^2 = x^2+y^2$

$x = r\cos\theta$

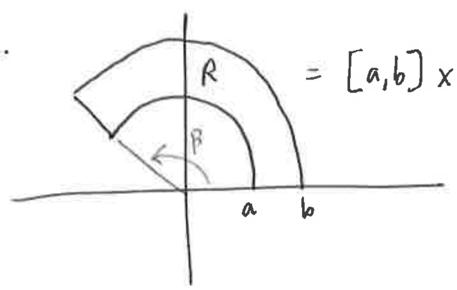
$y = r\sin\theta$

$\theta = \arctan\left(\frac{y}{x}\right)$

A "polar rectangle" is a portion of an annulus:

$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$

ex.

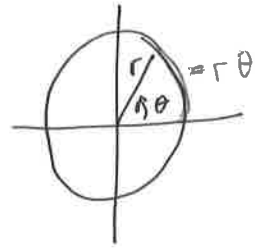


$= [a,b] \times [0,\beta]$

Not the best way to think of this.

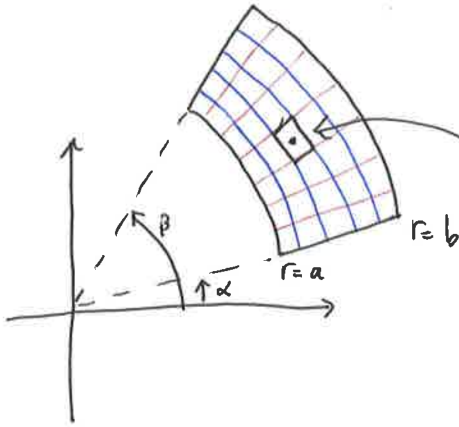
We need to know the area of this "rectangle". It will be equal to the sum of the arc lengths of the curves making up the annulus:

Arc length of a segment of a circle is $s = r\theta$



$$A = \int_a^b r\theta\, dr = \frac{1}{2}r^2\theta\Big|_a^b$$

Think of a region



partition the region

$R_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$

midpoint of R_{ij} is

$r_i^* = \frac{1}{2}(r_i + r_{i-1})$, $\theta_j^* = \frac{1}{2}(\theta_j + \theta_{j-1})$

the area of a sector of a circle is $\frac{1}{2} r^2 \theta$

$$\left.\begin{aligned} \Delta r_i &= r_i - r_{i-1} \\ \Delta \theta_j &= \theta_j - \theta_{j-1} \end{aligned}\right\} \quad \Delta A_{ij} = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1}) \qquad \text{(outer - inner)}$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})(\theta_j - \theta_{j-1})$$

$$= r_i^* \, \Delta r_i \, \Delta \theta_j$$

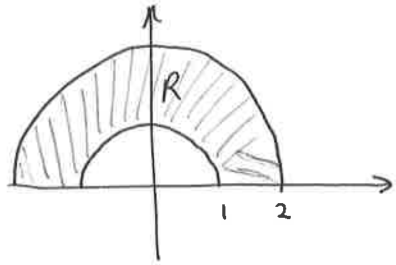
so that $\boxed{dA = r\,dr\,d\theta}$ in the limit.

Thus, we get the equation:

If f is a continuous fcn on the polar rect. $R = [a,b] \times [\alpha, \beta]$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Ex. Evaluate $\iint_R (3x + 4y^2)\, dA$ where is the region of the upper half plane bdd by $x^2+y^2=1$ and $x^2+y^2=4$



$1 \leq r \leq 2$
$0 \leq \theta \leq \pi$

$$f(x,y) = 3x + 4y^2 \Rightarrow f(r,\theta) = 3r\cos\theta + 4r^2\sin^2\theta$$

$$\iint_R f(x,y)\, dA = \int_0^{\pi} \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta \, dr \, d\theta$$

$$= \int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta \Big|_1^2 d\theta$$

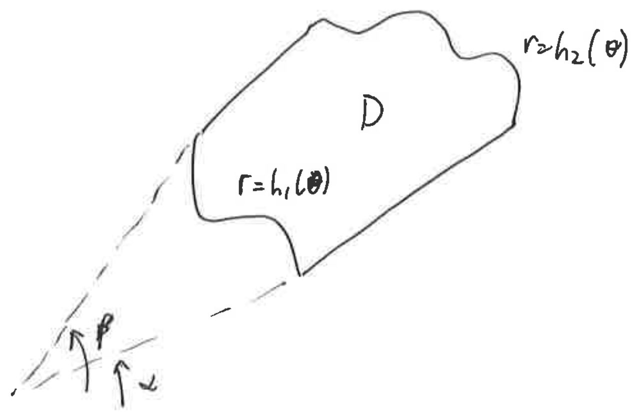
$$= \int_0^{\pi} 7\cos\theta + 15\sin^2\theta \, d\theta$$

$$= 7\sin\theta\Big|_0^{\pi} + \frac{15}{2}\int_0^{\pi} 1 - \cos 2\theta \, d\theta$$

$$= \frac{15}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= \boxed{\frac{15\pi}{2}}$$

We can extend this idea to more complicated domains



If h_1, h_2 are continuous functions of θ between α and β, then

$$\boxed{\iint_D f(r,\theta)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta}$$

As usual, taking $f=1$, we obtain the area of the domain

$$A(D) = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r\, dr\, d\theta$$

Ex. Find the area of the region bounded by 0 and $h(\theta)$

$$A = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, d\theta = \int_\alpha^\beta \frac{1}{2}\left[h(\theta)\right]^2 d\theta$$

The same formula we saw in Ch. 9.

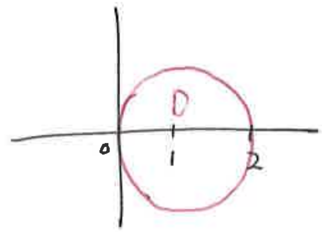
Ex. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$ above the xy-plane and inside the cylinder $x^2+y^2=2x$.

Draw the domain:

$$x^2 - 2x + y^2 = 0$$

$$(x^2 - 2x + 1) - 1 + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$



In polar coords this circle (the boundary) can be written as

$$r^2 = 2r\cos\theta \quad \text{or} \quad r = 2\cos\theta$$

This maps out the whole circle between $-\pi/2$ and $\pi/2$.

Then we have

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} 4\cos^4\theta \, d\theta$$

$$= 8\int_0^{\pi/2} \cos^4\theta \, d\theta = 2\int_0^{\pi/2} (1 + \cos 2\theta)^2 \, d\theta$$

$$= 2\int_0^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= 2\Big[\theta + \sin 2\theta\Big]_0^{\pi/2} + \int_0^{\pi/2} (1 + \cos 4\theta) \, d\theta$$

$$= 2\theta + 2\sin 2\theta + \theta + \frac{1}{4}\sin 4\theta \Big|_0^{\pi/2}$$

$$= \boxed{\frac{3\pi}{2}}$$