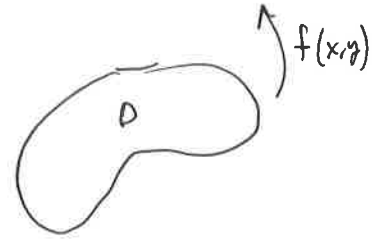
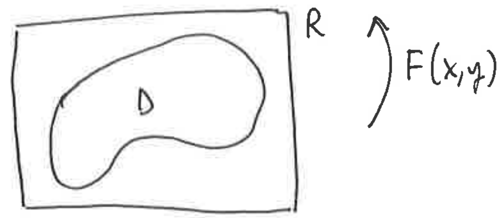
§12.2: Double Integrals over general regions

We want to integrate over regions like : 

and not just over rectangles.

The idea is to enclose the region D in a rectangle R :



and define a new function $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \text{ but } (x,y) \notin D. \end{cases}$

then $$\boxed{\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA}$$

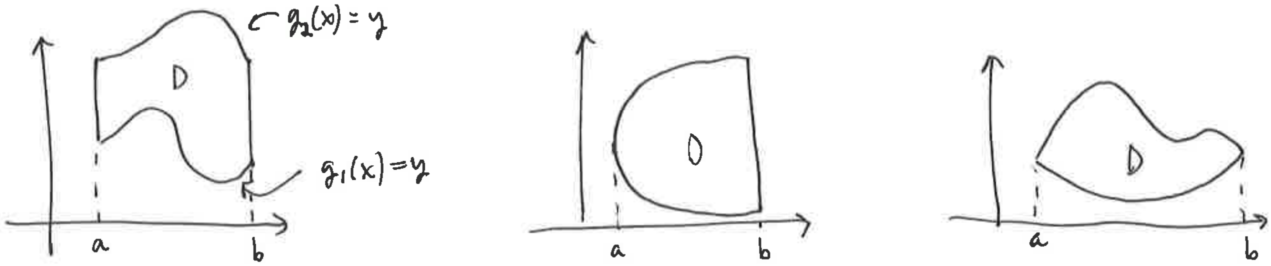
Notice that this idea makes sense no matter how we choose the rectangle R, as long as it contains all of D.

<u>Defn.</u> A plane region D is said to be of <u>Type I</u> iff

$$D = \{ (x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$$

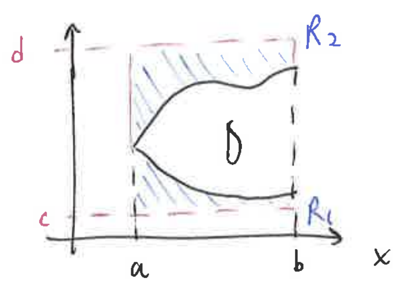
i.e., if it lies between the graphs of two continuous functions $g_1(x)$ and $g_2(x)$ on $[a,b]$.

Examples of Type I regions:



To evaluate a type I integral we choose a rectangle

$R = [a,b] \times [c,d]$ with $d \geq g_2(x)\ \forall x$, $c \leq g_1(x)\ \forall x$.



Then

$$\boxed{\iint_D f(x,y)\, dA} = \int_a^b \int_c^d F(x,y)\, dy\, dx$$

$$= \int_a^b \left[\int_c^d F(x,y)\, dy \right] dx$$

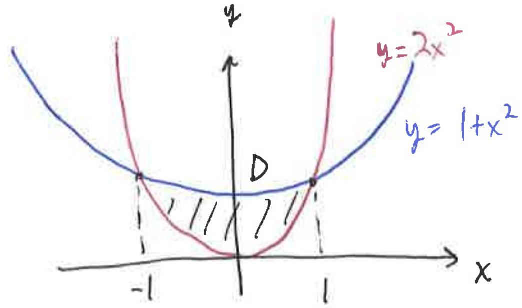
(*) $$= \int_a^b \left[\cancelto{0}{\int_c^{g_1(x)} F(x,y)\, dy} + \underbrace{\int_{g_1(x)}^{g_2(x)} F(x,y)\, dy}_{= \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy} + \cancelto{0}{\int_{g_2(x)}^{d} F(x,y)\, dy} \right] dx$$

$$= \boxed{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx}$$

For a continuous f on a type I region, we will calculate the integral by using this formula.

$\underline{Ex}$. Evaluate $\iint_D (x+2y)\, dA$ where D is bdd by $y = 2x^2$ and $y = 1+x^2$

First draw the region:



$x \in [-1,1]$

~~but the region is symmetric.~~

function is odd though!!

$$\iint_D (x+2y)\, dA = \iint_{\text{"}D\text{"}} (x+2y)\, dA$$

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2 \Big]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \Big[x(1+x^2) - 2x^3 + (1+x^2)^2 - 2^2x^4 \Big]\, dx$$

$$= \int_{-1}^{1} x + x^3 - 2x^3 + 1 + 2x^2 + x^4 - 4x^4\, dx$$

$$= \int_{-1}^{1} 1 + x + 2x^2 - x^3 - 3x^4\, dx$$

$$= \Big[x + \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{3}{5}x^5 \Big]_{-1}^{1}$$

$$= \left(1 + \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{1}{4} - \tfrac{3}{5}\right) - (\cdots) = \frac{(60+30+40-15-36) - (\cdots)}{60} = \frac{(130-51) - (\cdots)}{60}$$

$$= \frac{64}{30} = \boxed{\frac{32}{15}}$$

Type II regions are bounded by continuous functions $x = h_1(y)$ and $x = h_2(y)$ on the "left" and "right".

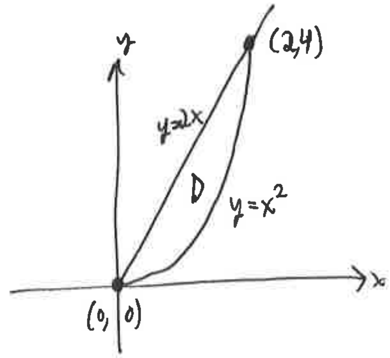
$$D = \{ (x,y) : h_1(y) \le x \le h_2(y), \; c \le y \le d \}$$

where h_1, h_2 are defined on $[c,d]$.

In this case we get:

$$\boxed{\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy}$$

Ex. $\iint_D x^2 + y^2 \, dA$ w/ D bdd between $y = 2x$ and $y = x^2$.



This D can be either a type I or type II region. Let's try it as type II.

"upper" bd: $x = \sqrt{y}$

lower bd: $x = \frac{1}{2} y$, $y \in [0,4]$

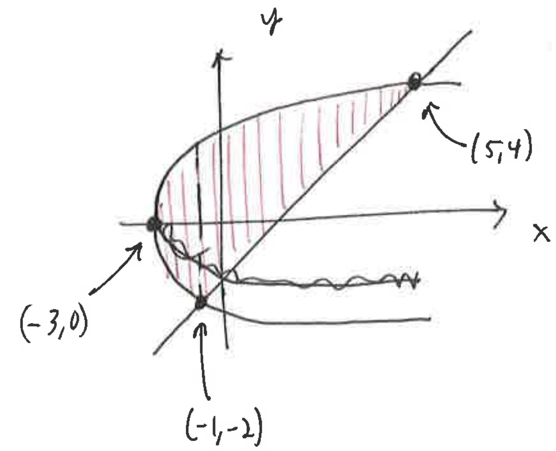
$$\iint_D f(x,y)\, dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} x^2 + y^2 \, dx\, dy$$

$$= \int_0^4 \left[x y^2 + \frac{1}{3} x^3 \right]_{\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 y^{5/2} - \frac{1}{2} y^3 + \frac{1}{3} y^{3/2} - \frac{1}{6} y^3 \, dy$$

$$= \int_0^4 \frac{4}{3} y^{3/2} - \frac{2}{3} y^3 \, dy = \left[\frac{2}{5} \cdot \frac{4}{3} y^{5/2} - \frac{2}{12} y^4 \right]_0^4 = \frac{8}{15}(32) - \frac{4^4}{6} = \boxed{\frac{216}{35}}$$

Ex. $\iint_D xy\, dA$ where D is bdd by $y = x-1$ and $y^2 = 2x+6$.

Draw the region:



To do this as a type I region we would have to break it into 2 pieces. Instead, we'll do it as type II.

$$\begin{cases} UB: & x = y+1 \\ LB: & x = \frac{1}{2}y^2 - 3 \\ \text{dom}: & y \in [-2, 4] \end{cases}$$

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy$$

$$= \int_{-2}^{4} \left[\frac{1}{2}x^2 y \right]_{\frac{1}{2}y^2-3}^{y+1} dy$$

$$= \int_{-2}^{4} \left[\frac{1}{2}(y+1)^2 y - \frac{1}{2}\left(\frac{1}{2}y^2-3\right)^2 y \right] dy$$

$$= \int_{-2}^{4} \left(\frac{1}{2}y^2 + y + \frac{1}{2}\right)y - \left(\frac{1}{8}y^4 + \frac{3}{2}y^2 - \frac{9}{2}\right)y \, dy$$

$$= \int_{-2}^{4} \frac{1}{2}y^3 + y^2 + \frac{1}{2}y - \frac{1}{8}y^5 + \frac{3}{2}y^3 - \frac{9}{2}y \, dy$$

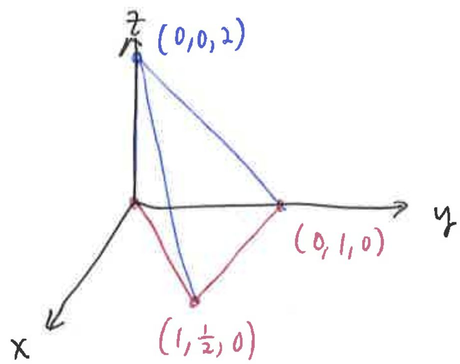
$$= \int_{-2}^{4} -4y + y^2 + 2y^3 - \frac{1}{8}y^5 \, dy$$

$$= \left[-2y^2 + \frac{1}{3}y^3 + \frac{1}{2}y^4 - \frac{1}{48}y^6 \right]_{-2}^{4} = \cdots = 36$$

Ex. Find the volume of the tetra-hedron bounded by

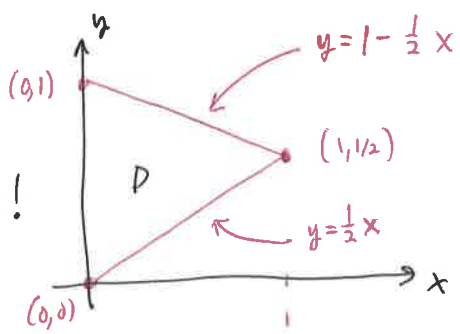
$$x + 2y + z = 2 \Rightarrow z = 2 - x - 2y$$
$$x = 2y$$
$$x = 0$$
$$z = 0$$

The solid:



The plane region:

Type I!



$$f(x,y) = 2 - x - 2y$$

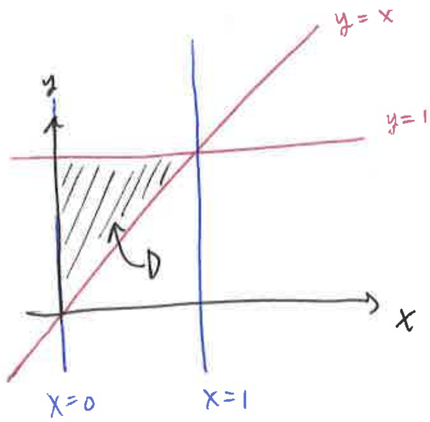
$$V = \iint_D f(x,y)\, dA = \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} 2 - x - 2y \; dy \; dx$$

FTIs: Evaluate it. (RE)

Ex. Evaluate: $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$

Problem: we have no idea what the antiderivative of $\sin(y^2)$ is.

Idea: Think of it as an area integral over a region



This D can be type I or type II.

The integral is written as type I, so we can try to rewrite it as type II.

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \left[x \sin y^2 \right]_0^y dy = \frac{1}{2}\int_0^1 2y \sin y^2\, dy$$

$u = y^2$

$du = 2y\, dy$

$$= \frac{1}{2}\int_0^1 \sin u\, du = \frac{1}{2}\left(-\cos u\right)\Big|_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}\cos 0$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}\cos 1}$$

☺

Some properties of Double Integrals:

Usual stuff:

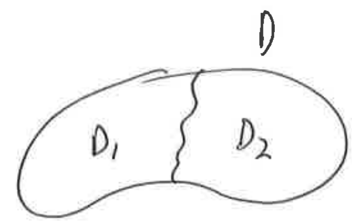
$$\iint_D f \pm g \, dA = \iint_D f \, dA \pm \iint_D g \, dA$$

$$\iint_D cf \, dA = c\iint_D f \, dA$$

If $f \leq g$ for all $(x,y) \in D$ then

$$\iint_D f \, dA \leq \iint_D g \, dA$$

If $D = D_1 \cup D_2$ w/ $D_1 \cap D_2 = \emptyset$,



then

$$\iint_D f \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA.$$

If $A(D)$ = area of D, then

$$\iint_D 1 \, dA = A(D) \qquad \text{(prove it!)}$$

If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ then

$$m\, A(D) \leq \iint_D f(x,y) \, dA \leq M\, A(D) \qquad \text{(prove it!)}$$