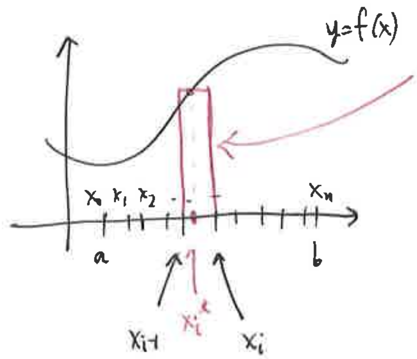
Chapter 12: Multiple Integrals

§ 12.1 Double Integrals over rectangles

As usual, we'll begin by flashing back to Calc I.

The definite integral



Area of box $= (x_i - x_{i-1}) \cdot f(x_i^*)$

$= f(x_i^*) \Delta x_i$

where x_i^* is any point in the interval $[x_{i-1}, x_i]$.

Then the area under $y=f(x)$ between a and b is <u>approximately</u>

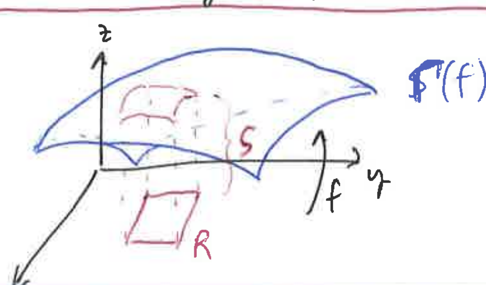
$$\sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

And <u>exactly</u>

(Riemann Sum) $\longrightarrow$ $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*) \Delta x_i$ if the limit exists.

This exact area is the definite integral of f from a to b.

Now we want to generalize this to consider volume under a surface. We will spend some time developing the details so that the idea is clear as we move forward. First, we want to find the <u>volume under a surface over a rectangle in the domain.</u>



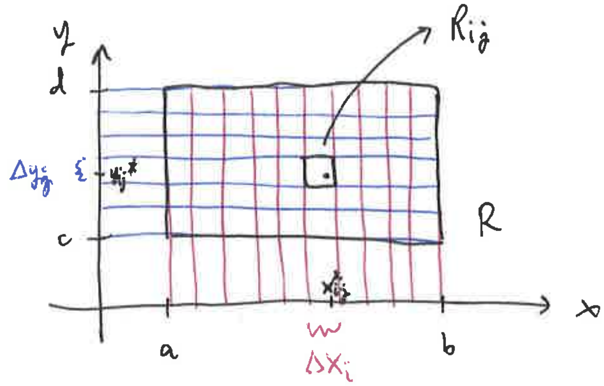
To simplify the situation at first, assume f is a positive function.

$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\} = [a,b] \times [c,d]$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R,\ 0 \le z \le f(x,y)\}$ solid "over" R

We want to find the volume of S.

Start by partitioning the domain R:



Partition $[a,b]$ into n intervals and $[c,d]$ in m intervals. This creates an array of $m \times n$ subrectangles of R.

$$\left.\begin{aligned} \Delta x_i &= x_i - x_{i-1} \\ \Delta y_j &= y_j - y_{j-1} \end{aligned}\right\} \quad \Delta A_{ij} = \Delta x_i \Delta y_j \text{ is the area of } R_{ij}.$$

We will estimate the volume under the surface above R_{ij} by a box (3D).
To estimate the height of the surface, we need to pick a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$ and find $f(x_{ij}^*, y_{ij}^*) = z_{ij}^*$.

Then the volume of S_{ij} is approximately $z_{ij}^* \Delta A_{ij} = f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$

Just like in Calc 1, we add up all of the volumes S_{ij} to approximate the volume of S:

$$\begin{aligned} \operatorname{Vol}(S) &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} \operatorname{Vol}(S_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\, \Delta A_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\, \Delta y_j\, \Delta x_i \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta x_i\, \Delta y_j \end{aligned}$$

As you can guess, we now find the exact volume of S by ~~thus~~ taking the limit as $\Delta x_i, \Delta y_j \to 0$ (or $n, m \to \infty$):

$$\mathrm{Vol}(S) = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_i \sum_j z_{ij}^* \, \Delta A_{ij}$$

<u>Definition</u>. The double integral of f over the rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

"Double Riemann Sum"

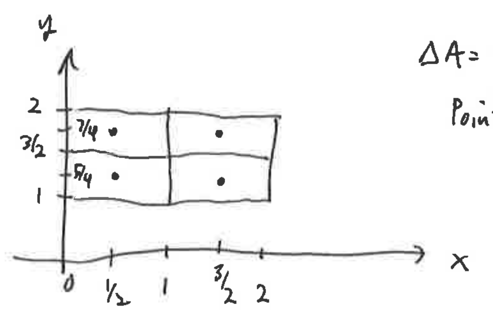
if the limit exists, where ΔA is fixed. (take a regular partition.)

A function is called <u>integrable</u> if this limit exists.

<u>Ex</u>. Use the midpoint rule to ~~evaluate~~ estimate

$$\iint_R (x - 3y^2)\, dA \quad \text{where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$$

w/ $m = n = 2$.



$\Delta A = (1)(\tfrac{1}{2}) = \tfrac{1}{2}$

Points: $(\tfrac{1}{2}, \tfrac{5}{4})$
$(\tfrac{1}{2}, \tfrac{7}{4})$
$(\tfrac{3}{2}, \tfrac{5}{4})$
$(\tfrac{3}{2}, \tfrac{7}{4})$

$$\iint_R (x-3y^2)\, dA \approx \frac{1}{2}\left[\frac{1}{2} - \frac{75}{16} + \frac{1}{2} - \frac{147}{16} + \frac{3}{2} - \frac{75}{16} + \frac{3}{2} - \frac{147}{16}\right]$$

$$= \frac{1}{2}\left[4 - \frac{444}{16}\right] = \boxed{\frac{-95}{8}} = -11.875$$

Iterated Integrals:

Recall that integrals are difficult to compute by the definition, but the evaluation thm (FTC part 2) is "very" simple (well... simpler).

Suppose that f is a continuous function of x and y on $R = [a,b] \times [c,d]$. We write $\int_c^d f(x,y)\,dy$ to mean that f is integrated ~~considered~~ as a function of y and x is held fixed.

This is called partial integration.

Now
$$A(x) = \int_c^d f(x,y)\,dy$$
is a function of only x.

We can now integrate A wrt x:
$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

We could do the same thing in the reverse order:
$$\int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Ex. Evaluate: $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ and $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

This leads us to:

<u>Fubini's Theorem.</u> If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx}$$

More generally, this is true if f is bdd on R, f is discontinuous on only a finite number of smooth curves, and the iterated integrals exist.

<u>Ex</u>. Evaluate $\iint_R (x - 3y^2)\,dA$ where $R = [0,2] \times [1,2]$.

get -12.

<u>Ex</u>. $\iint_R y \sin(xy)\,dA$, $R = [1,2] \times [0,\pi]$

Do $\iint y \sin(xy)\,dx\,dy$

The other way is much tougher.

Get 0.

<u>Ex</u>. In the special case when $f(x,y) = g(x)h(y)$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy.$$

Such a function f is called separable.

Ex. Evaluate $\iint_R \sin x \cos y \, dA$ where $R = [0, \pi/2] \times [0, \pi/2]$

$$= \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \left[-\cos \tfrac{\pi}{2} + \cos 0\right]\left[\sin \tfrac{\pi}{2} - \sin 0\right]$$

$$= [1][1] = 1$$

(25) Ex. $\iint_R xy e^{x^2 y} \, dA \qquad R = [0,1] \times [0,2]$

(26) Ex. $\iint_R \frac{x}{1+xy} \, dA \qquad R = [0,1] \times [0,1]$

* Integrating over general regions is a ~~other~~ generalization of this idea. We need to be careful about the region though.