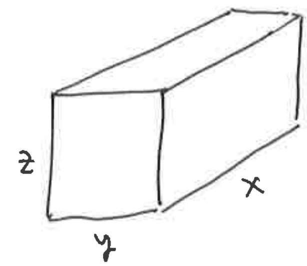
§11.8: Lagrange Multipliers

Example. A rectangular box w/ no top is to be made of $12 m^2$ of cardboard. Find the maximum ~~length~~ volume of such a box.



$$V = xyz$$

The surface area is given by

$$xy + 2xz + 2yz = 12 \qquad \text{(no top, remember)}$$

Solving this for z we get

$$xy + 2z(x+y) = 12$$

$$2z(x+y) = 12 - xy$$

$$z = \frac{12 - xy}{2(x+y)}$$

So that we can write $V(x,y) = xy\left(\frac{12 - xy}{2(x+y)}\right) = \frac{12xy - x^2y^2}{2(x+y)}$

$$D_xV = \frac{2(x+y)(12y - 2xy^2) - (12xy - x^2y^2)(2)}{4(x+y)^2} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2}$$

$$D_yV = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum then $D_xV = D_yV = 0$, but $x, y \neq 0$ since $V \neq 0$.

Thus we must solve $\begin{cases} 12 - 2xy - x^2 = 0 \\ 12 - 2xy - y^2 = 0 \end{cases}$

This implies that $x^2 = y^2$ so that $x = y$ (since $x, y > 0$).

Plugging $x=y$ into either equation yields

$$12 - 3x^2 = 0$$

$$\text{so that } x = y = \sqrt{4} = 2.$$

So $(2,2)$ is a critical point which must be a max in this problem. $V(2,2) = (2)(2)\dfrac{12-(2)(2)}{2(2+2)} = 4\left(\frac{8}{8}\right) = \boxed{4}\ m^3$.

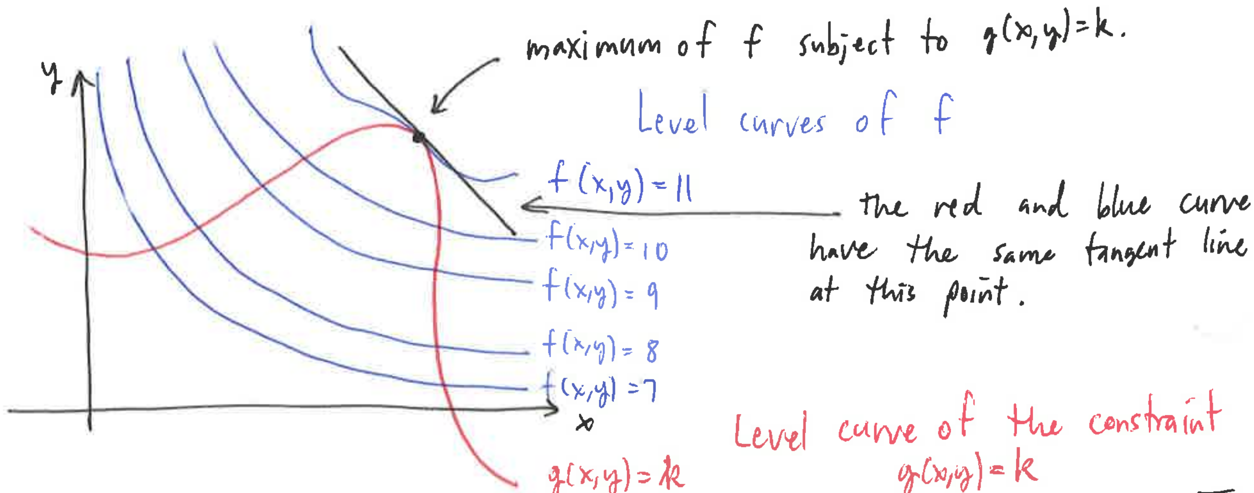
This was an example of maximizing a function $(V(x,y))$ subject to a constraint $(SA(x,y,z) = k = 12)$.

In general, we want to maximize or minimize $f(x,y,z)$ subject to $g(x,y,z) = k$.

In 2D:

We want to find extreme values of $f(x,y)$ subject to $g(x,y) = k$. That is, we want to find the extreme values of f along the level curve $g(x,y) = k$.

The situation looks like this:



The extrema occur when the level curves of f intersect the level curve $g=k$ and the two share the same tangent line. That is, when their gradient vectors at an intersection point are parallel: $\nabla f(a,b) = \lambda \nabla g(a,b)$ for some $\lambda \neq 0$.

This set up was all heuristic, but it is correct as long as we write it down carefully.

Thm. Lagrange Multipliers

To find the max and min of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, assuming they exist and $\nabla g \neq 0$ on the surface ~~the~~ $g(x,y,z)=k$,

1. Find all values of x, y, z, and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $g(x,y,z)=k$.

2. Evaluate f at all of the resulting points from part 1. The largest is the max. The smallest is the min.

Writing ∇f and ∇g in components we get a system of equations:

$$\begin{cases} D_x f = \lambda D_x g \\ D_y f = \lambda D_y g \\ D_z f = \lambda D_z g \\ g = k \end{cases}$$

This is a system of 4 eqns & 4 unknowns.

Example. Let's repeat the example of the box.

$$f(x,y,z) = V(x,y,z) = xyz$$

$$g(x,y,z) = 2xz + 2yz + xy = 12$$

$$\nabla f = \langle yz,\ xz,\ xy \rangle$$

and $\nabla g = \langle 2z+y,\ 2z+x,\ 2x+2y \rangle$

So our system looks like:

$$\begin{cases} 1.\quad yz = \lambda(2z+y) \\ 2.\quad xz = \lambda(2z+x) \\ 3.\quad xy = \lambda(2x+2y) \\ 4.\quad 2xz + 2yz + xy = 12 \end{cases}$$

Sometimes we need ingenuity to solve systems like this. We can solve each of 1., 2., 3. for λ, then set them equal.

$$\lambda = \frac{yz}{2z+y} = \frac{xz}{2z+x} = \frac{xy}{2x+2y}$$

we don't care what λ is exactly. We want x, y, z.

This yields:

$$yz(2z+x) = xz(2z+y) \qquad \cancel{2}y\cancel{z^2} + \cancel{xyz} = \cancel{2}x\cancel{z^2} + \cancel{xyz}$$

$$yz(2x+2y) = xy(2z+y) \qquad \cancel{2xyz} + 2y^2z = \cancel{2xyz} + xy^2$$

$$xz(2x+2y) = xy(2z+x) \qquad 2x^2z + \cancel{2xyz} = \cancel{2xyz} + x^2y$$

$$\Rightarrow \boxed{y = x}$$

$$2z = x \Rightarrow \boxed{z = \tfrac{1}{2}x}$$

$$2z = y$$

Plug these two into

$$2xz + 2yz + xy = 12$$

$$2x(\tfrac{1}{2}x) + 2x(\tfrac{1}{2}x) + xx = 12$$

$$\Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

but -2 is not physically possible, so $x = y = 2$ and $z = \frac{1}{2}x = 1$

$$\boxed{V(2,2,1) = 4}$$

One more example:

Ex. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and furthest from the point $(3,1,-1)$.

$$g(x,y,z) = x^2+y^2+z^2 = 4$$

$$f(x,y,z) = d^2(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

$$\nabla f = \langle 2(x-3), 2(y-1), 2(z+1) \rangle$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

So our system is:

$$\left.\begin{aligned} 2x-6 &= 2\lambda x \\ 2y-2 &= 2\lambda y \\ 2z+2 &= 2\lambda z \end{aligned}\right\} \quad \frac{2x-6}{2x} = \frac{2y-2}{2y} = \frac{2z+2}{2z}$$

$$x^2+y^2+z^2=4$$

$$\cancel{4xy} - 12y = \cancel{4xy} - 4x$$

$$\cancel{4xz} - 12z = \cancel{4xz} + 4x$$

$$\cancel{4yz} - 4z = \cancel{4yz} + 4y$$

$$\Rightarrow \left.\begin{aligned} x &= 3y \\ x &= -3z \end{aligned}\right\}! \quad \begin{aligned} y &= \tfrac{x}{3} \\ z &= \tfrac{-x}{3} \end{aligned}$$

$$y = -z$$

$$x^2 + \frac{x^2}{9} + \frac{x^2}{9} = 4$$

$$\frac{11}{9}x^2 = 4$$

$$x^2 = \frac{36}{11}$$

$$x = \pm\sqrt{\frac{36}{11}} = \pm\frac{6}{\sqrt{11}}$$

Then the points are $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$ and $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

min max

check those!