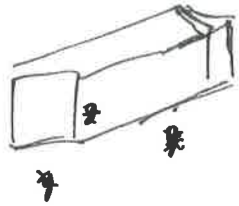
Ex. Volume of a box:



More than just two variables is ok :)

$V = xyz$

$dV = (yz)dx + (xz)dy + (xy)dz$

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§11.5: The Chain Rule

Case I. Let $z = f(x,y)$ be a differentiable function of x and y, and $x = x(t)$, $y = y(t)$ be smooth functions of t. Then z is a smooth function of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Chain Rule I.

Proof: ~~[illegible]~~ p. 625f. RE.

Idea: Write out Δz as in def'n of differentiability, divide everything by Δt, and take a limit. ▢

Ex. $z = x^2 y + 3xy^4$ where $x = \sin 2t$ and $y = \cos t$. Find $dz/dt|_{t=0}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

Now $x(0) = \sin(0) = 0$

$y(0) = \cos 0 = 1$

So $\left.\frac{dz}{dt}\right|_{t=0} = (\cancel{2(0)(1)} + 3(1)^4)(2(1)) + (0^2 + \cancel{12(0)(1)^3)(0)}$

$= 3 \cdot 2 = \underline{\underline{6}}$

Ex. P: pressure, V: volume, T: temperature

Ideal gas law: $PV = \underset{k}{\underline{8.31}}\, T$ (for a particular gas)

Find $\frac{dP}{dt}$ when $T = 300\ K$, $dT/dt = 0.1\ K/s$, $V = 100\ L$, $dV/dt = 0.2\ L/s$.

$$P = 8.31\frac{T}{V} = P(T,V) \quad P(300,100) = 8.31\left(\frac{300}{100}\right) = 24.93.$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$= \left(\frac{8.31}{V}\right)\frac{dT}{dt} + \left(-\frac{8.31\,T}{V^2}\right)\frac{dV}{dt}$$

$$= \left(\frac{8.31}{100}\right)(0.1) + \left(\frac{-8.31\,(300)}{10000}\right)(0.2)$$

$$= \cdots = -0.04155$$

So in this situation the pressure is decreasing at a rate of $\underline{0.042\ kPa/s.}$ (kilo Pascals per sec)

<u>Case II</u>. Now suppose that $z = f(x,y)$ is smooth, and $x = x(s,t)$ and $y = y(s,t)$ are smooth in s and t. Then z is smooth in s,t and

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}, \quad \text{and} \quad \boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}.$$

Ex. $z = e^x \sin y$, $x = st^2$, $y = s^2 t$. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

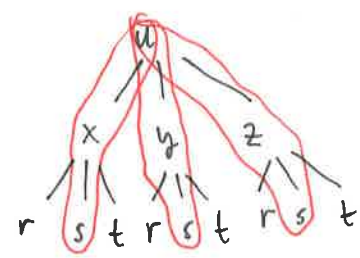
DO IT...

The General Chain Rule

Th. 4, p. 628.

Ex. If $u = x^4 y + y^2 z^3$, $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$. Find $\partial u / \partial s$ when $r = 2, s = 1, t = 0$.

Make a "tree diagram":



so
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial z}$$
$$= (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

Then at $r = 2, s = 1, t = 0$ we have

$$x = 2, \quad y = 2, \quad z = 0, \quad \text{and}$$

$$\frac{\partial u}{\partial s} = (4)(8)(2)(2) + (16 + 0)(4) + 0$$
$$= 2^7 + 2^6 = 128 + 64 = \underline{192}$$

Ex. $z = f(x(u(t,s), v(t,s)), y(u(t,s), v(t,s)))$

Find formulas for $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$. Do it.

Implicit Differentiation

Suppose $y = f(x)$ and $z = F(x,y)$, moreover that $F(x,y) = 0$.

Then $\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$.

Since $dx/dx = 1$, then we can solve for dy/dx as

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y} \qquad (*)$$

(bad notation, but easier to write down.)

as long as $\partial F/\partial y \neq 0$!

In Advanced Calculus (m 5??) you will prove a very important theorem called The Implicit Function Theorem.

It says:

If $F(a,b) = 0$, F is defined on a neighborhood of (a,b), $\partial F/\partial y \neq 0$, and $\partial F/\partial x$, $\partial F/\partial y$ are continuous on the disc, then the equation $F(x,y) = 0$ defines y as a function of x ~~and~~ near (a,b) (ie., on the disc in some sence) and $dy/dx = -\partial_x F/\partial_y F$.

This is an extremely important theorem in Differential Geometry (DG), the maths of choice of your fearless instructor ☺.

Ex. Find $y' = dy/dx$ if $x^3 + y^3 = 6xy$.

Write $F(x,y) = x^3 + y^3 - 6xy = 0$.

Then $$dy/dx = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-(3x^2 - 6y)}{(3y^2 - 6x)} = \frac{6y - 3x^2}{3y^2 - 6x}$$

* This is essentially just a new way of looking at an old idea. Let's generalize it to a function $z = f(x,y)$ now!

So, suppose $z = f(x,y)$ and $\exists$ a function $F(x,y,z) = 0$ that implicitly relates z, x, y.

Then the Chain rule gives us:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$ so as long as $\frac{\partial F}{\partial z} \neq 0$,

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}}$$

Similarly, $$\boxed{\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}}$$ Do it!

There is a version of the ImFT that tells us when such a $z = f(x,y)$ exists as well.

Ex. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Do it.