# § 11.2: Limits and Continuity

The limit of a function of $n$ variables is similar to a limit of a function of one variable. We'll write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

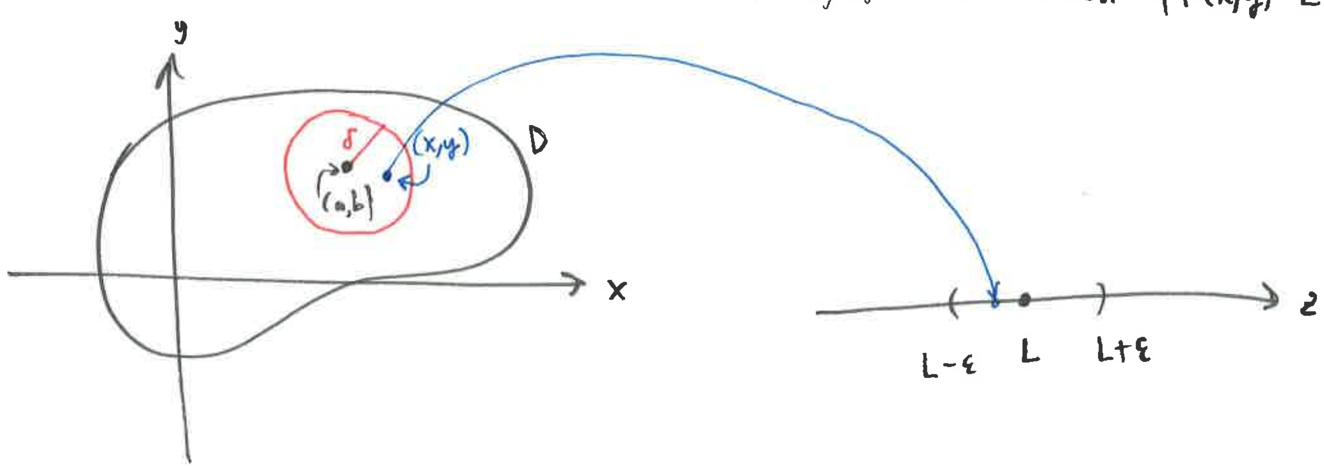
This means that the value of $f(x,y)$ approaches $L$ as we approach $(a,b)$ along any path in the domain of $f$.

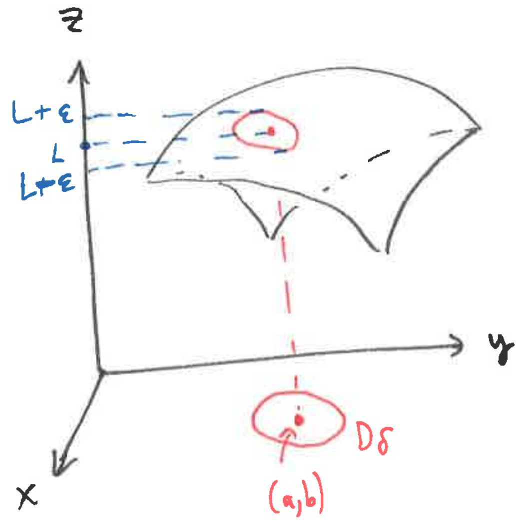
i.e., we can make $f(x,y)$ as close to $L$ as we want by choosing a pt $(x,y) \neq (a,b)$ in the domain of $f$ that is sufficiently close to $(a,b)$.

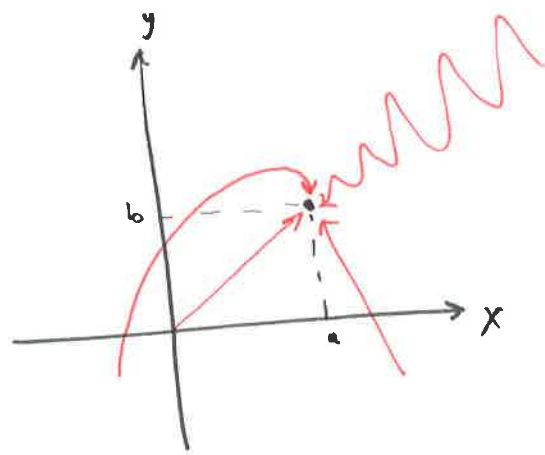
Def'n. Let $f$ be a function of two variables whose domain $D$ contains points arbitrarily close to $(a,b)$. Then we say that the limit of $f(x,y)$ as $(x,y)$ approaches $(a,b)$ is and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

iff for every number $\varepsilon > 0$ there is a corresponding $\delta(\varepsilon) > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.






We have to be able to approach $(a,b)$ along any path and get the same answer for the limit to exist.

(*) If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along one path, and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a different path, with $L_1 \neq L_2$, then the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Ex. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

First, approach along the x-axis, i.e., w/ pts $(x,0)$.

then $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$

Now, approach along the y-axis: $(0,y)$ to get

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1.$$

Not equal, so DNE.

Ex. For $f(x,y)=\dfrac{xy}{x^2+y^2}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Along x-axis:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

along $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Not equal, therefore DNE.

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Ex. $f(x,y)=\dfrac{xy^2}{x^2+y^4}$. Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

take paths $y=mx$ : $f(x,y)\to 0$

and $x=y^2$ : $f(x,y)\to \frac{1}{2}$

DNE

Ex. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ exists. Find it and show that it exists.

along x-axis we get $\lim\limits_{x\to 0} \dfrac{0}{x^2} = 0$. {Along other paths, we also get 0.}

To show that this is the limit we need to use the $\varepsilon$-$\delta$ defn.

Let $\varepsilon>0$. We need to find a $\delta=\delta(\varepsilon)>0$ such that

if $0<\sqrt{x^2+y^2}<\delta$, then $\left|\dfrac{3x^2y}{x^2+y^2}-0\right|<\varepsilon$

or $\dfrac{3x^2|y|}{x^2+y^2}<\varepsilon$.

Now, since $y^2 \geq 0$, then $x^2 \leq x^2+y^2$, so that $\frac{x^2}{x^2+y^2} \leq 1$, therefore

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}.$$

Thus, if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Therefore by the $\varepsilon$-$\delta$ definition of limit,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0. \qquad \square$$

As you can see, it takes a lot of work to show that a limit does exist.

## Continuity

Defn. A function $f(x,y) = z$ is continuous at $(a,b)$ iff

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say that $f$ is continuous on $D$ if $f$ is continuous at every point $(a,b) \in D$.

Intuitively: A surface that is the graph of a continuous function has no holes or breaks. It also has no "asymptotic singularities" (like infinite cusps or cones).

A polynomial function of two variables is a linear combination (sum or difference) of terms like $x^m y^n$, $m, n$ ~~positive~~ nonneg. integers.

ex. $f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$

A rational function is a quotient of polynomials.

ex. $g(x,y) = \dfrac{2xy+1}{x^2+y^2}$

Thm. Polynomials are continuous. Rational functions are continuous where they are defined (on D).

Ex. $\displaystyle\lim_{(x,y)\to(1,2)} \left(x^2y^3 - x^3y^2 + 3x + 2y\right)$

$= 1^2 2^3 - 1^3 2^2 + 3(1) + 2(2)$

$= 8 - 4 + 3 + 4 = 11$

Ex. where is $f(x,y) = \dfrac{x^2 - y^2}{x^2+y^2}$ continuous?

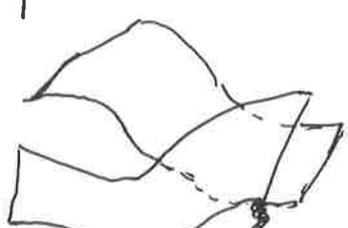
Everywhere except $(0,0)$. $D = \mathbb{R}^2 - \{\vec{0}\}$.

Ex. let $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Now $g$ is defined at $(0,0)$, but still not continuous.
The limit doesn't exist (last class).

Ex. $h(x,y) = \arctan\left(\frac{y}{x}\right)$ where is this continuous?

Everywhere except $x=0$. The graph looks like two sheets of tangent graphs.



or something...

Limits of functions of multiple variables.

Let $\vec{x} = \langle x_1, x_2, x_3, \dots, x_n \rangle$ be an n-vector in $\mathbb{R}^n$.

let $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ be a fixed point. (thought of as a position vector.)

$$d(\vec{x}, \vec{a}) = \|\vec{x} - \vec{a}\| = \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + \dots + (x_n-a_n)^2}$$

Let $f : \mathbb{R}^n \to \mathbb{R} : \vec{x} \longmapsto f(\vec{x})$.

We say $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ iff

$\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that for all $\vec{x} \in D$ w/ $0 < \|\vec{x} - \vec{a}\| < \delta$,

$|f(\vec{x}) - L| < \varepsilon$.

If $\vec{x}, \vec{a}$ are 2-vectors, then this is the same defn as before.

A function $f(\vec{x})$ is continuous at $\vec{a}$ iff

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$