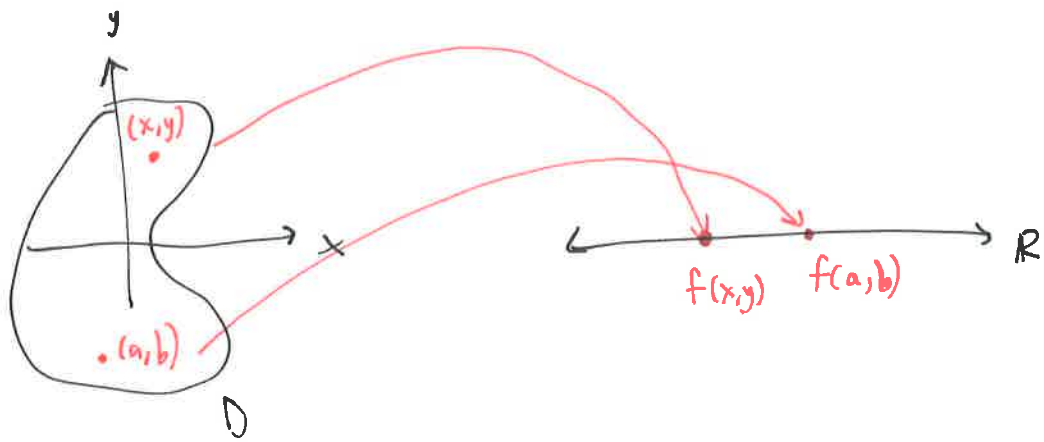
§11.1 : Functions of Several Variables

Def^n. A function of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D (the domain) a unique real number $f(x,y)$.

The range is the set of real numbers

$$R = \{ f(x,y) \mid (x,y) \in D \}.$$



As usual, if no domain is specified, then the domain is all pairs (x,y) where the function makes sense.

Ex. Find the domain of $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

$x \neq 1$

and $x + y + 1 \geq 0$

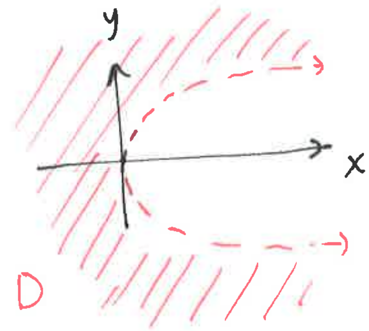
or $\Rightarrow x \geq -1-y$

So $D = \{ (x,y) : x \geq -1-y \text{ and } x \neq 1 \}$

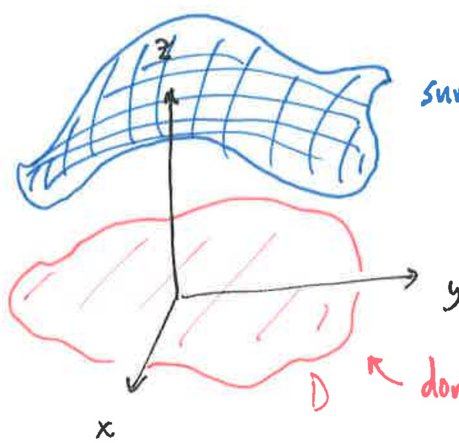
Ex. Domain of $f(x,y) = x\ln(y^2 - x)$

$$y^2 - x > 0$$
$$\Rightarrow \quad x < y^2$$

So $D = \{(x,y) \mid x < y^2\}$.



Defn. If f is a function of two variables w/ domain D, then the graph of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$, and $(x,y) \in D$.



surface in $\mathbb{R}^3$ above D.

domain in $\mathbb{R}^2$

Ex. Sketch the graph of $f(x,y) = 6 - 3x - 2y$

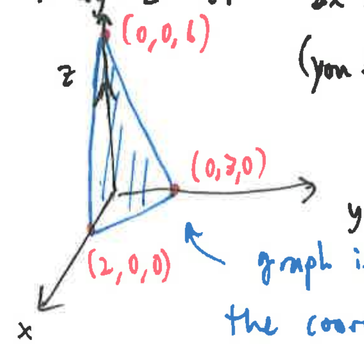
Idea: Find the intercepts of $6 - 3x - 2y = z$ or $3x + 2y + z = 6$

(you should know what this is now!!)

If $(x,y) = (0,0)$ then $z = 6$

If $(x,z) = (0,0)$ then $y = 3$

If $(y,z) = (0,0)$ then $x = 2$



graph is a plane that cuts the coord axes like this.

A function of the form $f(x,y) = z = ax + by + c$ is called a <u>linear function.</u> Its graph is always a plane.

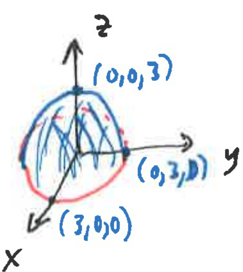
<u>Ex</u>. $g(x,y) = \sqrt{9 - x^2 - y^2}$ graph it.

$$z = \sqrt{9 - x^2 - y^2}$$

$$\Rightarrow z^2 = 9 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 + z^2 = 9$$ Sphere centered at origin w/ radius 3.

<u>BUT</u>: It's only the top half of the sphere. why?



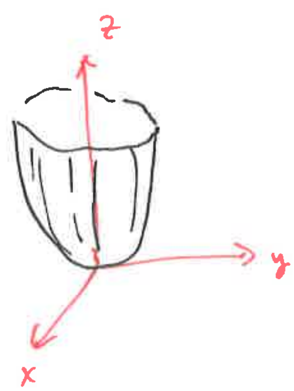
... or something...

<u>Ex</u>. $h(x,y) = 4x^2 + y^2$

$$z = 4x^2 + y^2$$ Domain $= \mathbb{R}^2$.

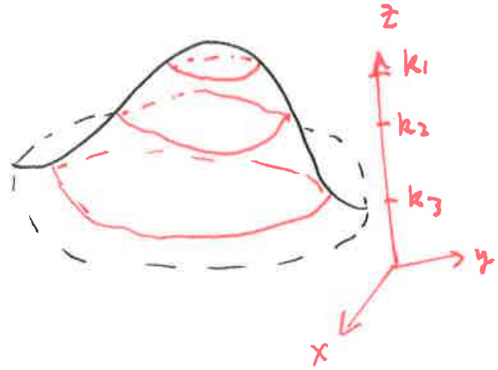
$$\frac{z}{4} = x^2 + \frac{y^2}{4}$$

This is a quadratic surface: <u>Elliptic paraboloid.</u>



__Defn__. The __level curves__ of a function $z = f(x,y)$ are the curves satisfying the equation $f(x,y) = k$, where $k \in \text{Range}(f)$.

Think of slicing the surface $z = f(x,y)$ with planes parallel to the xy-plane, at a height of $z = k$.



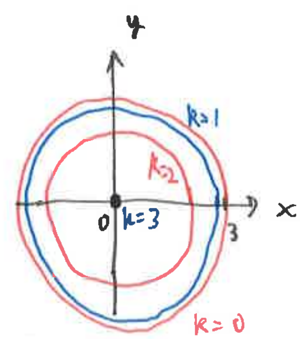
level curves are also called __contour curves.__

__Common example of level curves:__ topographical maps of altitudes on earth.

__Ex.__ Sketch the level curves of $g(x,y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$

$k = 0$: $0 = \sqrt{9 - x^2 - y^2}$

$x^2 + y^2 = 9$



$k = 1$: $1 = \sqrt{9 - x^2 - y^2}$

$1 = 9 - x^2 - y^2$

$x^2 + y^2 = 8$

$k = 3$: $3 = \sqrt{9 - x^2 - y^2}$

$9 = 9 - x^2 - y^2$

$x^2 + y^2 = 0.$

$k = 2$: $2 = \sqrt{9 - x^2 - y^2}$

$4 = 9 - x^2 - y^2$

$5 = x^2 + y^2$

Examples of graphs of functions together with their level curves on page 597.

Functions of three of more variables:

We can extend the definition of a function of 2 variables so that the domain is a region in $\mathbb{R}^n$, or specifically $\mathbb{R}^3$.

Ex. Find the domain of $f(x,y,z) = \ln(z-y) + xy \sin z$

$$z - y > 0 \Rightarrow z > y$$

Thus $D = \{(x,y,z) \mid z > y\}$, the half space of all points lying above the plane $z = y$.

Ex. Find level surfaces of the function $f(x,y,z) = x^2 + y^2 + z^2$

They are $x^2 + y^2 + z^2 = k$, i.e., spheres of radius $\sqrt{k}$ centered at the origin.

Notation: Consider a function of n variables $f(x_1, \ldots, x_n)$. We can consider this as a vector function (in the sense that if takes in vectors in its domain) and write

$$f(x_1, \ldots, x_n) = f(\vec{x}) \quad \text{where } \vec{x} = \langle x_1, \ldots, x_n \rangle.$$