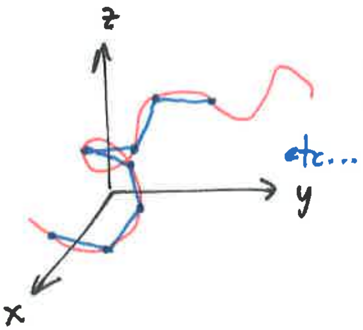
Geometry of Curves in Space : 10.8 & 10.9



γ or C : space curve - graph of vector function.

Suppose the curve is parametrized:

$$\vec{r}(t) = c(t) = \langle x(t), y(t), z(t) \rangle$$

since curves and vector functions are so closely related, we confuse them.

Recall from Chapters 7/9: the arc length of a 2D-curve is given by:

$$L(c|[a,b]) = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In 3D, the formula is the same:

$$L(c|[a,b]) = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}\, dt = \int_a^b \|\dot{c}(t)\|\, dt$$

Ex. $c(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$

Find the arc length from $(1,0,0)$ to $(1,0,2\pi)$

i.e., $t: 0 \to 2\pi$

~~$\|\dot{c}(t)\| = \sqrt{\cos^2 t + \sin^2 t + 0}$~~

$$\dot{c}(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + 1\, \vec{k}$$

$$\|\dot{c}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$L(c|[a,b]) = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\pi\sqrt{2}}$$

Recall that a single curve c can be parametrized in many different ways. Some are more useful than others.

Also recall the arc length function:

$$s(t) = \int_a^t \|\dot{c}(u)\|\, du \qquad \text{(via the FTC)}$$

s gives the arc length from a set point $c(a)$ to the point $c(u)$.

Also by the FTC: $\quad \dfrac{d}{dt} s(t) = \dfrac{d}{dt}\displaystyle\int_a^t \|\dot{c}(u)\|\, du = \|\dot{c}(t)\|$

or $\quad \dot{s}(t) = \|\dot{c}(t)\|$.

It's often useful to parametrize by arc length.

$s(t)$ is an increasing one-to-one function, so we can solve for t as a function of s: $t = t(s)$. Then do a change of variables to write $c = c(t(s))$.

Then, if $s=3$ $\;c(t(3)) = \vec{r}\,(t(3))$ is the position vector of the point 3 units of arc length away from the starting point of the curve.

Ex. Reparametrize the helix $c(t) = \langle \cos t, \sin t, t \rangle$ by arclength. w/ initial point $(1,0,0)$, in the direction of increasing t.

From last Ex, $\dot{s}(t) = \| \dot{c}(t) \| = \sqrt{2}$

so $s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$

then $t = \frac{s}{\sqrt{2}}$

The reparametrized curve is then

$$c(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle .$$ Word.

Curvature:

Recall the unit tangent vector field on a curve $c(t) = \vec{r}(t)$:

$$T(t) = \frac{\dot{c}(t)}{\| \dot{c}(t) \|}$$

indicates the direction of curve at that t-value.

Notice: T changes very slowly if c is nearly straight.
T changes more quickly if c is very "curvy".

Thus, we define the curvature of c as

$$\kappa = \left\| \frac{dT}{ds} \right\| = \frac{\| \dot{T} \|}{\| \dot{c} \|}(t) = \kappa(t).$$

Ex. Show that the curvature of a circle w/ radius a is $1/a$.

$c(t) = \langle a\cos t, a\sin t\rangle$

$\dot{c}(t) = \langle -a\sin t, a\cos t\rangle$

$\|\dot{c}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a| = a$

$T(t) = \frac{\dot{r}}{\|\dot{r}\|} = \langle -\sin t, \cos t\rangle$

$\dot{T} = \langle -\cos t, \sin t\rangle$

$\|\dot{T}\| = \sqrt{\cos^2 t + \sin^2 t} = 1.$

$$\boxed{\text{Thus, } \kappa(t) = \frac{\|\dot{T}\|}{\|\dot{r}\|} = \frac{1}{a}.}$$

Theorem: The curvature of a curve c w/ pos. function $\vec{r}$ is

$$\kappa(t) = \frac{\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3} = \frac{\|\dot{c}(t) \times \ddot{c}(t)\|}{\|\dot{c}(t)\|^3}$$

Proof: RE. Follow along w/ work in book. p. 573. $\square$

Ex. $\vec{r}(t) = \langle t, t^2, t^3\rangle$ Find κ at $\langle 0,0,0\rangle$.

$\dot{\vec{r}}(t) = \langle 1, 2t, 3t^2\rangle$

$\ddot{\vec{r}}(t) = \langle 0, 2, 6t\rangle$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 12t^2 - 6t^2, 0 - 6t, 2\rangle = \langle 6t^2, -6t, 2\rangle$

$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{36t^4 + 36t^2 + 4}$ $\Rightarrow \kappa(0) = \frac{\sqrt{4}}{1^{3/2}} = \underline{2}.$

$\|\dot{\vec{r}}\|^3 = (1 + 4t^2 + 9t^4)^{3/2}$

Normal and binormal vectors:

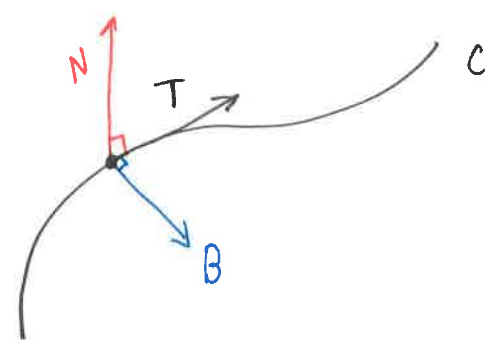
Recall $\|T(t)\| = 1$ for all t, and $T(t) \cdot \dot{T}(t) = 0$. (Ex from last class.)

This means $\dot{T} \perp T$ for all t.

$\dot{T}$ is not necessarily a unit vector itself, but

$$N(t) = \frac{\dot{T}(t)}{\|\dot{T}(t)\|} \quad \text{is.}$$

$N(t)$ is the principal unit normal vector of $c = \vec{r}(t)$.



Two vectors determine a plane, which then has a normal vector, the binormal vector to the curve:

$$B(t) := T(t) \times N(t)$$

Together $\{T, N, B\}$ form a frame or basis for $\mathbb{R}^3$ centered at the point on the curve c.

Def'n. The plane determined by N and B is called the normal plane to C at P. It contains all vectors orthogonal to C.

The plane determined by T and N is called the osculating plane of C at P. It "kisses" the curve at P. The plane is "tangent" to the curve.

Ex. Find the normal and binormal vectors for the helix:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\dot{\vec{r}}(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{2}$$

$$T(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

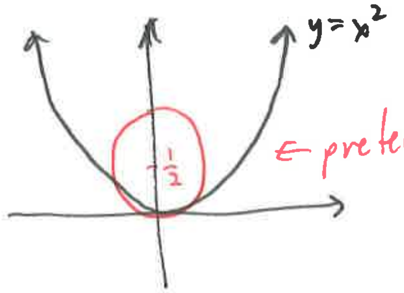
$$\dot{\vec{T}}(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

$$\|\dot{\vec{T}}(t)\| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

so, $$\boxed{N(t) = \frac{\dot{T}}{\|\dot{T}\|} = \langle -\cos t, -\sin t, 0 \rangle}$$

$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \boxed{\frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle = B}$$

Another Def'n. The circle that lies in the osculating plane, has the same tangent to c at P, lies on the concave side of c (toward N) and has radius $1/\kappa = \rho$ is called the osculating circle of c at P.



← pretend that is osculating.

Curves as paths of particles: Motion in space

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position function of a particle moving in space.

The velocity is then

$$\vec{v}(t) := \dot{\vec{r}}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle,$$

speed is

$$\|\vec{v}(t)\| = \|\dot{\vec{r}}(t)\| = \frac{ds}{dt},$$

and acceleration is

$$\cancel{\vec{a}(t) = \|\dot{\vec{v}}(t)\| = \|\ddot{\vec{r}}(t)\|} \qquad \vec{a}(t) := \dot{\vec{v}}(t) = \ddot{\vec{r}}(t).$$

whoops!!

$\underline{\text{Ex}}$. $\vec{r}(t) = \langle t^3, t^2 \rangle$ Find $\vec{v}$, $\|\vec{v}\|$, $\vec{a}$ when $t=1$.

$$\begin{cases} \vec{v} = \dot{\vec{r}} = \langle 3t^2, 2t \rangle \\ \vec{v}(1) = \langle 3, 2 \rangle \\ \|\vec{v}\| = \sqrt{9+4} = \sqrt{13} \\ \vec{a} = \dot{\vec{v}} = \langle 6t, 2 \rangle \\ \vec{a}(1) = \langle 6, 2 \rangle \end{cases}$$

$\underline{\text{Ex}}$. $\vec{a} = \langle 4t, 6t, 1 \rangle$ Find $\vec{r}(t)$.

$\vec{v}(0) = \langle 1, -1, 1 \rangle$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 4t, 6t, 1 \rangle\,dt$$

$$= \langle 2t^2, 3t^2, t \rangle + \vec{c}$$

$$\vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{c} \Rightarrow \vec{c} = \langle 1, -1, 1 \rangle$$

so $\vec{v}(t) = \langle 2t^2+1, 3t^2-1, t+1 \rangle$

$$\vec{r} = \int \vec{v}\,dt = \int \langle 2t^2+1, 3t^2-1, t+1 \rangle\,dt$$

$$= \langle \tfrac{2}{3}t^3+t, t^3-t, \tfrac{1}{2}t^2+t \rangle + \vec{c}$$

$$\vec{r}(0) = \langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{c} \Rightarrow \vec{c} = \langle 1, 0, 0 \rangle$$

so $\boxed{\vec{r}(t) = \langle \tfrac{2}{3}t^3+t+1, t^3-t, \tfrac{1}{2}t^2+t \rangle}$

Newton's Second Law of motion:

$$\vec{F}(t) = m\,\vec{a}(t) \quad \text{where } m \text{ is the mass of the particle.}$$

HW may use this.

<u>Tangential and normal acceleration</u>:

let speed be $\sigma(t) := \|\vec{v}(t)\|$ (book uses v, but that's confusing.)

$$\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{\vec{v}(t)}{\sigma(t)} \quad \text{or just } \frac{\vec{v}}{\sigma}$$

Thus $\underline{\vec{v} = \sigma\vec{T}} \Rightarrow \vec{a} = \dot{\vec{v}} = \sigma'\vec{T} + \sigma\dot{\vec{T}} \quad (*)$

Recall $\kappa = \dfrac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \dfrac{\|\dot{\vec{T}}\|}{\sigma}$, so $\|\dot{\vec{T}}\| =$ [redacted] $\kappa\sigma$

Also, $\vec{N} = \dfrac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|}$, so $\dot{\vec{T}} = \|\dot{\vec{T}}\|\,\vec{N} = \kappa\sigma\vec{N}$

Then $(*)$ becomes

$$\vec{a}(t) = \underbrace{\sigma'}_{\text{tangential}}\vec{T} + \underbrace{\kappa\sigma^2}_{\text{normal}}\vec{N}$$

For simplicity we write: $\vec{a} = a_T\vec{T} + a_N\vec{N}$.

It turns out (see book pp. 582 f)

$$\boxed{a_T = v' = \frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{\|\dot{\vec{r}}\|}}$$ and

$$\boxed{a_N = \kappa v^2 = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|}}$$

Fill in details, if time.

Ex. A particle moves w/ position function $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$

Find a_T and a_N.

$$a_T = \frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{\|\dot{\vec{r}}\|}$$

$\dot{\vec{r}} = \langle 2t, 2t, 3t^2 \rangle$

$\ddot{\vec{r}} = \langle 2, 2, 6t \rangle$

$\dot{\vec{r}} \cdot \ddot{\vec{r}} = 4t + 4t + 18t^3 = 8t + 18t^3$

$\|\dot{\vec{r}}\| = \sqrt{4t^2 + 4t^2 + 9t^4} = \sqrt{8t^2 + 9t^4}$

$$\Rightarrow \boxed{a_T = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}}$$

$$a_N = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|}$$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 12t^2 - 6t^2, 6t^2 - 12t^2, 0 \rangle = \langle 6t^2, -6t^2, 0 \rangle$

$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{36t^4 + 36t^4} = \sqrt{72t^4} = \sqrt{72}\, t^2$

$$\Rightarrow \boxed{a_N = \frac{\sqrt{72}\, t^2}{\sqrt{8t^2 + 9t^4}}}$$

Chapter 10: Grand Finale

Kepler's Laws of Planetary Motion:

1. A planet revolves around the sun in an elliptical orbit w/ the sun at one focus.

2. The line joining the Sun to a planet sweeps out equal areas in equal times.

3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Extra Credit Project: Kepler project from book. Up to 2% on Final Grade. Due by last day of class (9 May). Will be graded slightly harder than normal projects.