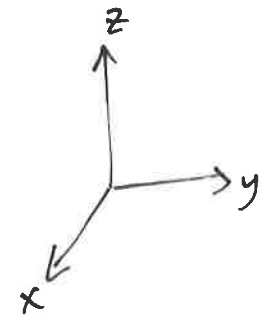
Review of Sections 10.1 – 10.4

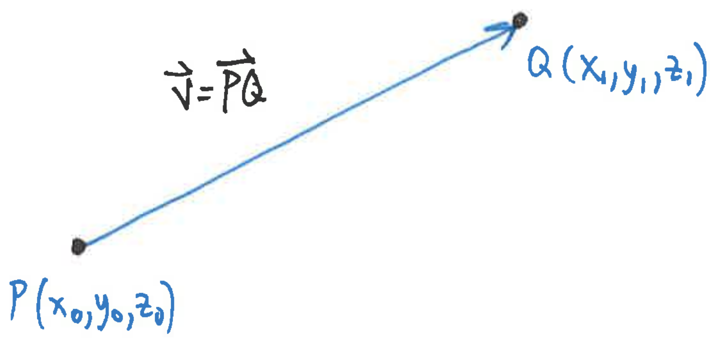
Three-dimensional space $\mathbb{R}^3$

Points in $\mathbb{R}^3$ are represented by ordered triples: (x, y, z)



Vectors:

A vector is a directed line segment connecting two points in $\mathbb{R}^n$. ($n = 2$ or 3 for us.)



The vector $\vec{PQ}$ has initial point P and terminal point Q.

A vector is said to be in standard position if $P = \vec{0}\,(0,0,0)$.

In this case, we can completely describe $\vec{v} = \vec{PQ}$ by its terminal pt $Q(x_1, y_1, z_1)$. We write $\vec{v} = \langle x_1, y_1, z_1 \rangle$.

This is called the component form of $\vec{v}$.

We also frequently write $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in comp. form.

Standard unit vectors in $\mathbb{R}^3$ are:

$$\left.\begin{aligned} \vec{i} &= \langle 1, 0, 0 \rangle \\ \vec{j} &= \langle 0, 1, 0 \rangle \\ \vec{k} &= \langle 0, 0, 1 \rangle \end{aligned}\right\} \text{ so } \vec{v} \text{ can be written as } \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

All of the vectors in $\mathbb{R}^3$ form a vector space V^3. We are allowed to add two vectors, but we can't multiply them by each other.

$$\vec{v} = \langle v_1, v_2, v_3 \rangle, \quad \vec{u} = \langle u_1, u_2, u_3 \rangle, \quad k \in \mathbb{R}$$

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

verify: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$.

$$k\vec{v} = k\langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

so $\quad -\vec{v} = \langle -v_1, -v_2, -v_3 \rangle = (-1)\vec{v}$

$$\vec{v} - \vec{u} = \vec{v} + (-1)\vec{u} = \langle v_1 - u_1, v_2 - u_2, v_3 - u_3 \rangle$$

etc...

The length or magnitude of a vector is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad \text{or} \quad \|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

Pythagorean Theorem ish.

The Dot Product:

$$\vec{u} = \langle u_1, u_2, u_3 \rangle, \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

verify:

$$\begin{cases} \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \\ (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v}) = k(\vec{u} \cdot \vec{v}) \end{cases}$$

Moreover,

$$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} \quad \text{or} \quad \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$$\vec{0}\cdot\vec{v} = 0 \leftarrow \text{zero number}$$

zero vector

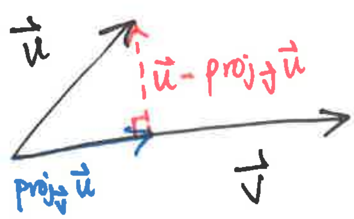
[Two vectors are said to be orthogonal if $\vec{u}\cdot\vec{v}=0$

~~This leads to~~ a formula to measure angles between two vectors:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

$$\Rightarrow \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

Projections



$$\text{proj}_{\vec{v}}\vec{u} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}}_{\#}\,\vec{v} \leftarrow \text{vector}$$

scaling factor

so all together it's a vector.

Recall that $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector (length = 1), and $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$. We can rewrite the projection as

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\,\frac{\vec{v}}{\|\vec{v}\|}$$

$\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ is a number called the scalar projection, or component projection of $\vec{u}$ onto $\vec{v}$. We write

$$\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

The Cross Product:

Only exists for vectors in V^3. (!)

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\,(u_2v_3 - v_2u_3) - \vec{j}\,(u_1v_3 - v_1u_3) + \vec{k}\,(u_1v_2 - v_1u_2)$$

$$= \langle u_2v_3 - v_2u_3,\ v_1u_3 - u_1v_3,\ u_1v_2 - v_1u_2 \rangle$$

A vector orthogonal to both $\vec{u}$ and $\vec{v}$.

* Right hand rule and picture.

There is also a trig formula for the magnitude of cross product:

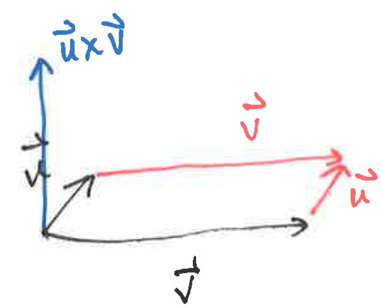
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta .$$

Fact: $\vec{u} \times \vec{0} = \vec{0}$

Two vectors are parallel iff $\vec{u} \times \vec{v} = \vec{0}$.

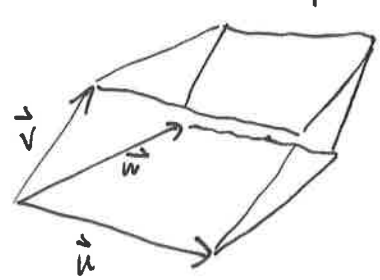
(Notice, this means that $\vec{0}$ is orthogonal and parallel to all other vectors in V^3.)

Geometrically:
The length of the cross product is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$



Triple Products and more geometry:

$\|\vec{v} \cdot (\vec{u} \times \vec{w})\|$ = volume of parallelepiped



(or something...)

* Review all Recommended Exercises From these Four sections!!