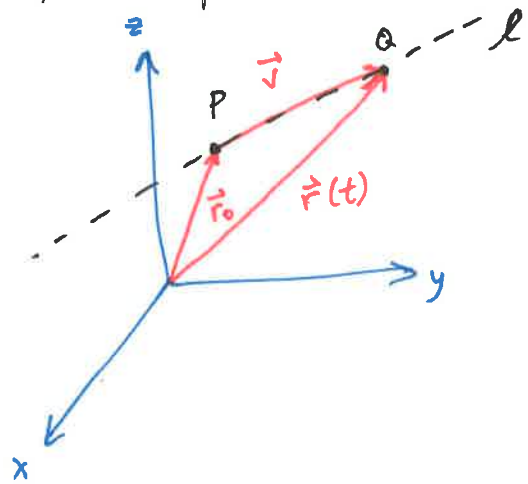
Sections 10.5 and 10.6

10.5: Equations of lines and planes in $\mathbb{R}^3$.

A line is determine by two points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$.



The point P determines a vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ in standard position. The two points P and Q determine a vector $\vec{v} = \overrightarrow{PQ}$ called the direction vector of the line.

The line ℓ is the described by

$$\ell: \; \vec{r}(t) = \vec{r}_0 + t\vec{v}$$

where $t \in \mathbb{R}$ is a parameter (cf. Ch.9), and $\vec{v} = \langle a, b, c \rangle = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

In coordinates this becomes:

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$
$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$\vec{r}$ is the position vector of a "particle" traveling along ℓ.

This is the vector equation of ℓ.

Parametric equations for ℓ are obtained by separating the components of $\vec{r}$:

$$\begin{cases} x = x(t) = x_0 + t\,\text{[redacted]}\,a \\ y = y(t) = y_0 + t\,\text{[redacted]}\,b \\ z = z(t) = z_0 + t\,\text{[redacted]}\,c \end{cases}$$

Solving each of these for t, then setting them equal, we obtain Symmetric equations for ℓ:

$$\frac{x - x_0}{\text{[redacted]}\,a} = \frac{y - y_0}{\text{[redacted]}\,b} = \frac{z - z_0}{\text{[redacted]}\,c}$$

~~Note: In the book the author uses~~ [redacted]

Ex. Find eq'ns of the line through $(6,1,-3)$ and $(2,4,5)$

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$$\vec{r}_0 = \langle 6, 1, -3 \rangle \qquad \vec{v} = \langle 4, -3, -8 \rangle$$

$$\vec{r}(t) = \langle 6 + 4t,\ 1 - 3t,\ -3 - 8t \rangle \quad , \ldots \text{ etc.} \ldots$$

Two lines are parallel if their direction vectors are parallel, and the lines do not coincide.

Two lines are perpendicular if their direction vectors are orthogonal, and the lines intersect.

Two lines are skew if they are not parallel and do not intersect.

Ex. Determine whether the lines are parallel, skew, perpendicular, or
[18] intersecting.

$$\ell_1: \ \vec{r}_1(t) = \langle 1+2t,\ 3t,\ 2-t \rangle$$

$$\ell_2: \ \vec{r}_2(s) = \langle -1+s,\ 4+s,\ 1+3s \rangle$$

$\vec{v}_1 = \langle 2, 3, -1 \rangle \qquad \vec{v}_2 = \langle 1, 1, 3 \rangle$

clearly not parallel.

$$\begin{cases} 1+2t = -1+s \\ 3t = 4+s \\ 2-t = 1+3s \end{cases}$$

$s = 3t-4$

$\Rightarrow 1+2t = -1+3t-4$

$\boxed{6 = t}$ from x-comp.

$\Rightarrow 2-t = 1+3(3t-4)$

$2-t = 1+9t-12$

$13 = 10t$

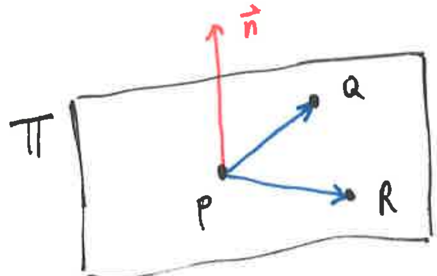
$\boxed{t = 13/10}$ from z-comp

$3/10 \neq 6$, so

the lines are <u>skew</u>.

<u>Planes:</u>

A plane in R^3 is determined by 3 points (not collinear, of course).



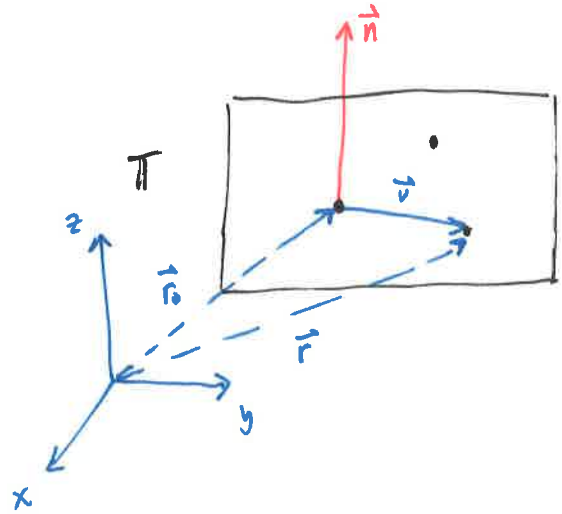
$\vec{n} = \vec{PQ} \times \vec{PR}$

is a <u>normal</u> <u>vector</u>

to the plane Π.

$\vec{n}$ is $\perp$ to Π.

Since $\vec{n} \perp \Pi$, then $\vec{n} \cdot \vec{v} = 0$ for any $\vec{v}$ in Π.



$\vec{v} = \vec{r} - \vec{r}_0$

$$\begin{cases} \vec{r} = \langle x, y, z \rangle \\ \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \\ \vec{n} = \langle a, b, c \rangle \end{cases}$$

so Π is completely determined by the equation:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$$

$$ax + by + cz = \boxed{ax_0 + by_0 + cz_0}$$ just a #, call it $-d$.

(−d bc the book does..)

$$\boxed{ax + by + cz = -d}$$

or

$$* \quad \boxed{ax + by + cz + d = 0}$$

Ex. Find an eq'n of plane w/ pts $P(1,3,2)$, $Q(3,-1,6)$, $R(5,2,0)$.

$$\boxed{6x + 10y + 7z = 50}$$

Two planes Π_1 and Π_2 intersect in a line unless they are parallel. The angle between two planes is the angle between their normal vectors.

<u>Ex</u>. $x+y+z=1$, $x-2y+3z=1$.

1. Find θ: $\arccos\left(\frac{2}{\sqrt{42}}\right) \approx 1.257$

2. Find equation of line ℓ of intersection:

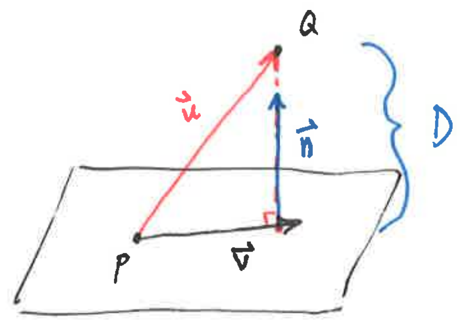
   a. find a pt on the line: set $z=0$, solve system, get $(1,0,0)$.

   b. find direction:

   $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 5,-2,-3\rangle$.

   So $\vec{r}(t) = \langle 1+5t, -2t, -3t\rangle$:)

Distance between a point and a plane.



$P(x_0, y_0, z_0)$

$Q(x_1, y_1, z_1)$

$\vec{n} = \langle a,b,c\rangle$

D is the length of the projection of $\vec{u}$ onto $\vec{n}$.

i.e.,

$$D = \left| \frac{\vec{u}\cdot\vec{n}}{\|\vec{n}\|} \right| = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \boxed{\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Ex. Find the distance between $(2,8,5)$ and $x-2y-2z=1$

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$$D = \frac{|2(1)+8(-2)+5(-2)-1|}{\sqrt{1^2+2^2+2^2}} = \frac{|2-16-10-1|}{3} = \boxed{\frac{25}{3}}$$