Math 344: Calculus III
Chapter 13 Notes

30 April – 9 May 2013

These notes are meant to be a guide to help us move more quickly through the end of the semester. You are expected to follow along in class, then reread them on your own, as necessary. I plan on doing most of the problems labeled as **Ex** in class, but any that we skip will automatically become recommended exercises. Speaking of recommended exercises, there are a few included in these notes. They are labeled as **RE**.

13.1: Vector Fields

A hand-written version of this section's notes is on my web page.

13.2: Path Integrals

We started this section but didn't finish it.

Let's start now by reviewing what we've already done. Remember that our text book calls path integrals "line" integrals, even though the paths we are interested in are rarely actual lines.

A *path* C is the image of a *parametrized curve* $\boldsymbol{r}(t) = \langle x(t), y(t), \ldots \rangle$, $a \leq t \leq b$; *i.e.,* it's just the image of the curve, disregarding the parametrization. Given a function $f : C \to \mathbb{R}$, we want to find the integral of f along C. This is a single integral called the *path integral of f along C*. We write

$$\int_C f(x,y)\, ds, \tag{1}$$

where ds is the *arc length element* from chapters 7, 9, and 10. Now recall that the arc length function for a path C is given by

$$s(t) = \int_a^t \|\dot{\boldsymbol{r}}(u)\|\, du, \tag{2}$$

where $\boldsymbol{r}(t)$ is any parametrization of C. Then

$$\frac{ds}{dt} = \frac{d}{dt}\int_a^t \|\dot{\boldsymbol{r}}(u)\|\, du = \|\dot{\boldsymbol{r}}(t)\| \tag{3}$$

by the Fundamental Theorem of Calculus. Thus we can write $ds = \|\dot{\boldsymbol{r}}(t)\|\, dt$, and the integral in equation (1) becomes

$$\int_C f(x,y), ds = \int_a^b f(x(t), y(t))\, \|\dot{\boldsymbol{r}}(t)\|\, dt = \int_a^b f(\boldsymbol{r}(t))\, \|\dot{\boldsymbol{r}}(t)\|\, dt, \tag{4}$$

for any parametrization $\boldsymbol{r}(t)$ of the path C. This is the formula that we will actually use to calculate path integrals. It's just a regular single integral from Calculus 1!

Recall that a directed line segment from p to q can be written as

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_p + t\boldsymbol{r}_q, \quad 0 \le t \le 1, \tag{5}$$

where $\boldsymbol{r}_p$ and $\boldsymbol{r}_q$ are the position vectors for p and q. You will probably need this to do your homework. (We used it in an example last class, and we'll use it again today.)

Path Integrals in Space

The only real difference here is that the function f is now a function of three variables $f = f(x, y, z)$, and the path C is given by a space curve $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$. The path integral equation now becomes

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}\,dt \\ &= \int_a^b f(\boldsymbol{r}(t))\,\|\dot{\boldsymbol{r}}(t)\|\,dt. \end{aligned} \tag{6}$$

Notice that the last part (the vector equation) is exactly the same as it was in 2D. It will always be the same, for every $\mathbb{R}^n$ and $\mathbb{V}^n$, no matter what n is. Therefore, this is the equation that you should remember!

Ex 1 Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C is given by two line segments. The first goes from $(2, 0, 0)$ to $(3, 4, 5)$, and the second goes from $(3, 4, 5)$ to $(3, 4, 0)$.

Ex 2 Evaluate $\int_C y \sin z \, ds$ where C is the circular helix defined by $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \leq t \leq 2\pi$.

Path Integrals Over Vector Fields

Let $\mathbf{F} = \mathbf{F}(x, y, z)$ be a continuous vector field on $\mathbb{R}^n$. (Think: force field.) If $\mathbf{T}$ is the unit tangent vector field along a path C, then the work done by $\mathbf{F}$ on a particle that moves along C is approximately $\mathbf{F} \cdot \mathbf{T}$ for small neighborhoods around each point. The idea is the usual one: partition the path C into small sub-paths, find $\mathbf{F} \cdot \mathbf{T}$ on each one, add them all up, then take a limit as the number of partition points goes to infinity; *i.e,* integrate. Then the work done by $\mathbf{F}$ is

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds. \tag{7}$$

But $\mathbf{T} = \dot{\boldsymbol{r}} / \|\dot{\boldsymbol{r}}\|$ and $ds = \|\dot{\boldsymbol{r}}\| \, dt$, so the equation becomes

$$\begin{aligned} W &= \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big) \cdot \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle \, dt \\ &= \int_a^b \mathbf{F}(\boldsymbol{r}(t)) \cdot \dot{\boldsymbol{r}}(t) \, dt \\ &=: \int_C \mathbf{F} \cdot d\boldsymbol{r}, \end{aligned} \tag{8}$$

where $\boldsymbol{r}(t)$ is any parametrization of the path C, as usual.

Ex 3 Find $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ where $\mathbf{F} = \langle xy, yz, zx \rangle$ and C is given by $\boldsymbol{r} = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

Ex 4 Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$. Show that

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Now we know why we were interested in so-called partial path integrals last class.

13.3: The Fundamental Theorem for Path Integrals

One half of the Fundamental Theorem of Calculus from Calculus 1 is the Evaluation Theorem. It says that if $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}F(x)\,dx = F(b) - F(a), \tag{9}$$

where F is any antiderivative of f; *i.e.,* $f(x) = F'(x)$ for all $x \in [a, b]$. We expect a similar theorem for path integrals.

Let $\mathbf{F}$ be a conservative vector field along a smooth path C. Remember, this means that there exists a function f such that $\mathbf{F}(\boldsymbol{r}(t)) = \nabla f(\boldsymbol{r}(t))$ for all $t \in [a, b]$, where $\boldsymbol{r}(t)$ is any parametrization of C. If ∇f is continuous along C, then

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)). \tag{10}$$

RE 5 Prove this theorem by reducing it to the Calculus 1 Evaluation Theorem.

Ex 6 Find the work done by gravity on a moving particle with mass m that travels from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along any piecewise-smooth curve C.

Independence of Path

We did an example last class which showed us that path integrals depend on the paths you integrate along, not just the starting and ending points. However, equation (10) and the last **Ex** suggest something different.

Clearly, if $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ does not depend on the path C. The situation appears to be slightly more general.

Theorem 7 *$\int_C \mathbf{F} \cdot d\boldsymbol{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\boldsymbol{r} = 0$ for every closed path C in D.*

Ex 8 Proof:

As we mentioned before, we already know that conservative vector fields are independent of path, essentially by definition. The next theorem tells us that conservative vector fields are the *only* ones that are independent of path.

Theorem 9 *Suppose D is an open, connected region in $\mathbb{R}^n$. Let $\mathbf{F}$ be a continuous vector field on D. If $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D;* i.e, *there is a function $f : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.*

RE 10 Read the proof: p. 745.

One question still remains: How can we tell if a vector field is conservative or not? The answer in (or on?) $\mathbb{R}^2$ comes in the form of two more theorems.

Theorem 11 *If* $\mathbf{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ *is a conservative vector field, where* P *and* Q *have continuous first-order partial derivatives on a domain* D*, then throughout* D *we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

RE 12 Prove this by using Clairaut's Theorem.

Theorem 13 *Let* $\mathbf{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ *be a vector field on an open, simply connected region* D*. Suppose that* P *and* Q *have continuous first-order derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D*; then* $\mathbf{F}$ *is conservative.*

We'll prove this theorem in the next section.

Ex 14 Determine whether or not the vector field $\mathbf{F}(x,y) = (x-y)\boldsymbol{i} + (x-2)\boldsymbol{j}$ is conservative.

Ex 15 Determine whether or not the vector field $\mathbf{F}(x, y) = (3+2xy)\boldsymbol{i}+(x^2-3y^2)\boldsymbol{j}$ is conservative.

Ex 16 (a) If $\mathbf{F}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$, find a potential function f such that $\mathbf{F} = \nabla f$. (b) Evaluate the path integral $\int_C \mathbf{F} \cdot d\boldsymbol{r}$, where C is determined by $\boldsymbol{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \leq t \leq \pi$.

Conservation of Energy

Newton's Second Law says that $\mathbf{F} = m\boldsymbol{a}$. Remembering that $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$ and $\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t)$, we get a second order differential equation (or SODE):

$$\mathbf{F}(\boldsymbol{r}(t)) = m\ddot{\boldsymbol{r}}(t).$$

Let C be a path defined by the parametrized curve $\boldsymbol{r}(t)$, $t \in [a, b]$. Furthermore, let $\boldsymbol{r}(a) = A$ and $\boldsymbol{r}(b) = B$.

RE 17 Use this and the fact that kinetic energy is given by $K(\boldsymbol{r}(t)) = \frac{1}{2}m\|\boldsymbol{v}(t)\|^2$ to show that

$$W = \int_C \mathbf{F} \cdot d\boldsymbol{r} = K(B) - K(A).$$

RE 18 Suppose that $\mathbf{F}$ is a conservative vector field; *i.e.,* $\mathbf{F} = \nabla f$. The *potential energy* of an object at a point (x, y, z) is defined to be $P(x, y, z) = -f(x, y, z)$ (hence the name *potential function*). Therefore $\mathbf{F} = -\nabla P$. Show that

$$P(A) + K(A) = P(B) + K(B).$$

This is called the *Law of Conservation of Energy.*

$**$ **Project 5:** On your own paper, work out all of the details for **RE**'s **5**, **12**, **17** and **18** from these notes.

Due: Tuesday, 14 May 2013, 8.00 am

13.4: Green's Theorem

In this section we will study the relationship between path integrals around a simple closed path C and double integrals over the simply connected domain D that C encloses. We'll start by stating the (in)famous *Jordan Curve Theorem.*

Theorem 19 *Every simple closed curve in* $\mathbb{R}^2$ *divides the plane into exactly two connected regions, one that is bounded* (the interior)*, and one that is not bounded* (the exterior).

This theorem is "infamous" because, although it seems obvious, it turned out to be extremely hard to prove. In fact the original proof that was presented by Camille Jordan (whom the theorem is named after) turned out to be flawed. Eventually, after many others had tried and failed, Oswald Veblen was able to finally give a rigorous proof. Even today, some mathematicians disagree over who should get credit for the first correct proof. Don't worry, we won't try to prove it here. In honor of this theorem, we frequently call simple closed curves *Jordan curves.*

Let C be a Jordan curve that encloses a domain D. By convention, we will say that C has *positive orientation*, or C is *positively oriented*, if we traverse C in the counter-clockwise direction. Therefore, the domain D is always on the left hand side, or driver's side (in the US, at least), of the path.

Draw some pictures here:

Theorem 20 (Green's Theorem.) *Let C be a positively oriented, piecewise smooth Jordan curve in the plane, and D the domain enclosed by it. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ has continuous partial derivatives on an open region containing D, then*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{11}$$

A notation frequently used in physics and applied math to denote the boundary of a domain D is ∂D. In this case $\partial D = C$, so we could rewrite the equation in the theorem as

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{12}$$

to emphasize the relationship that this theorem provides between a domain and its boundary.

RE 21 Read through the proof on p. 752f carefully, then do exercise 28, p. 757.

Ex 22 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Ex 23 Let $\mathbf{F} = (3y - e^{\sin x})\,\boldsymbol{i} + (7x + \sqrt{y^4+1})\,\boldsymbol{j}$. Evaluate $\int_C \mathbf{F}\cdot d\boldsymbol{r}$, where C is the circle $x^2+y^2 = 9$.

Both of these examples use Green's Theorem to convert a path integral to a double integral, but sometimes it's useful to use Green's Theorem in the other direction.

Ex 24 The area of a domain D is given by $\iint_D \, dA$. Use Green's Theorem to find at least 3 different path integral formulas for the area of D.

Ex 25 Find the area enclosed by the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Ex 26 Let

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

Show that $\int_C \mathbf{F} \cdot d\boldsymbol{r} = 2\pi$ for every positively oriented Jordan curve that encloses the origin.

RE 27 Use Green's Theorem to prove Theorem 13 when C is a Jordan curve. Can you think of a way to prove it for *any* closed curve?