Math 344: Calculus III
Final Exam

14 May 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such (no rounded off decimals). Each problem is worth 10 points.

1. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t)\rangle$, $0 \le t \le 1$.

$$\dot{\vec{r}} = \langle 3\sqrt{t}, -2\sin(2t), 2\cos(2t)\rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{9t + 4}$$

$$s = \int_0^1 \sqrt{9t+4}\, dt \qquad u = 9t+4 \quad u(1) = 13$$

$$du = 9\,dt \quad u(0) = 4$$

$$= \frac{1}{9}\int_4^{13} u^{1/2}\, du = \frac{1}{9}\cdot\frac{2}{3}\, u^{3/2}\Big|_4^{13} = \boxed{\frac{2}{27}\left(13^{3/2} - 8\right)}$$

2. A particle starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k}$. Find its position function.

$$\vec{v} = \int \vec{a}\, dt = \int 6t\vec{i} + 12t^2\vec{j} - 6t\vec{k}\, dt$$

$$= 3t^2\vec{i} + 4t^3\vec{j} - 3t^2\vec{k} + \vec{c}$$

$$\vec{v}(0) = \vec{0} + \vec{c} = \langle 1, -1, 3\rangle$$

$$\text{so} \quad \vec{v}(t) = \langle 3t^2 + 1, 4t^3 - 1, -3t^2 + 3\rangle$$

$$\vec{r} = \int \vec{v}\, dt = \langle t^3 + t + C_1, t^4 - t + C_2, -t^3 + 3t + C_3\rangle$$

$$\vec{r}(0) = \langle 0, 0, 0\rangle$$

$$\Rightarrow \boxed{\vec{r}(t) = \langle t^3 + t, t^4 - t, -t^3 + 3t\rangle}$$

3. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.

$2(2-t) - (1+3t) + 4t = 2$

$4 - 2t - 1 - 3t + 4t = 2$

$-t = -1$

$\Rightarrow t = 1$

so the pt of intersection is:

$(x(1), y(1), z(1)) = \boxed{(1,4,4)}$

4. Find the gradient of the function $f(x, y, z) = z^2 e^{x\sqrt{y}}$.

$$\boxed{\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}},\ \frac{+z^2 x}{2\sqrt{y}}\,e^{x\sqrt{y}},\ 2z\,e^{x\sqrt{y}} \right\rangle}$$

5. Find the directional derivative of $f(x,y) = 2\sqrt{x} - y^2$ at the point $(1,5)$ in the direction toward the point $(4,1)$.

$\vec{v} = \langle 4-1, 1-5 \rangle = \langle 3, -4 \rangle$

$\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$\nabla f = \langle -\frac{1}{\sqrt{x}}, 2y \rangle$

$\nabla f(1,5) = \langle -1, 10 \rangle$

$D_{\vec{u}} f(1,5) = \nabla f(1,5) \cdot \vec{u} = \langle -1, 10 \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$= \frac{-3}{5} - \frac{40}{5} = \boxed{\frac{-43}{5}}$

6. Prove the theorem: *Suppose f is a differentiable function of at least two variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $\|\nabla f(\mathbf{x})\|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

$D_{\vec{u}} f(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \vec{u} = \|\nabla f(\vec{x})\| \, \|\vec{u}\| \cos\theta = \|\nabla f(\vec{x_0})\| \cos\theta$

$\cos\theta$ is max when $\theta = 0$, thus

max of $D_{\vec{u}} f(\vec{x_0})$ occurs when $\vec{u}$ is in direction of $\nabla f(\vec{x})$, and max value equals $\|\nabla f(\vec{x})\|$. $\square$

7. Find the volume of the solid bounded above by the cone $z = \sqrt{x^2+y^2}$, bounded below by the plane $z = 0$, and sitting above the disk $x^2 + y^2 \leq 9$.

$$V = \iint_D z \, dA = \int_0^{2\pi} \int_0^3 r \; r \, dr \, d\theta \quad \text{in polar coords.}$$

$$= \int_0^{2\pi} d\theta \int_0^3 r^2 \, dr$$

$$= 2\pi \left(\frac{27}{3} - 0 \right) = \boxed{18\pi}$$

8. Evaluate the double integral

$$\iint_D \frac{y}{1+x^2} \, dA$$

where D is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$.

Type I.

$$= \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} \, dy \, dx = \int_0^1 \left. \frac{\frac{1}{2} y^2}{1+x^2} \right|_0^{\sqrt{x}} dx =$$

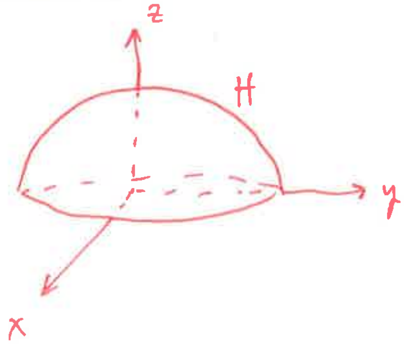
$$= \int_0^1 \frac{x}{1+x^2} \, dx = \left. \frac{1}{2} \ln(1+x^2) \right|_0^1 = \frac{1}{2} \ln(2) - \frac{1}{2} \ln(1)$$

$$= \boxed{\frac{1}{2} \ln(2)}$$

9. Evaluate the triple integral

$$\iiint_H z^2(x^2+y^2+z^2)\,dV$$

where H is the solid hemisphere that lies above the xy-plane, has center at the origin, and radius 2.



$H:\quad \rho: 0 \to 2$

$\theta: 0 \to \pi$

$\varphi: -\frac{\pi}{2} \to \frac{\pi}{2}$

$z^2 = \rho^2 \cos^2\varphi$

$x^2 + y^2 + z^2 = \rho^2$

$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

$$\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{\pi}\int_0^2 \rho^6 \cos^2\varphi \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \underbrace{\int_{-\frac{\pi}{2}}^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi}\ \underbrace{\int_0^{\pi} d\theta}_{=\pi}\ \underbrace{\int_0^2 \rho^6 \, d\rho}_{=\frac{2^7}{7}} = \boxed{0}$$

$$= \int_{-\frac{\pi}{2}}^{0} \cos^2\varphi \sin\varphi \, d\varphi + \int_0^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi$$

$$= -\int_0^1 u^2 \, du - \int_1^0 u^2 \, du$$

$$= -\int_0^1 u^2 \, du + \int_0^1 u^2 \, du = 0$$

10. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2+y^2=4$ with positive orientation.

$$\frac{\partial Q}{\partial x} = -y^2 \qquad \frac{\partial P}{\partial y} = x^2$$

$$\int_C x^2y\,dx - xy^2\,dy = \iint_D -y^2-x^2\,dA \qquad \text{polar coords}$$

$$\int_0^{2\pi}\int_0^2 -r^2\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \left(-\int_0^2 r^3\,dr\right)$$

$$= 2\pi\left(-\tfrac{1}{4}2^4+0\right) = \boxed{-8\pi}$$

11. Show that $\mathbf{F}(x,y) = (1+xy)e^{xy}\,\mathbf{i} + (e^y + x^2e^{xy})\,\mathbf{j}$ is conservative, then find a function f such that $\mathbf{F} = \nabla f$.

$$\frac{\partial Q}{\partial x} = 2xe^{xy} + x^2ye^{xy} \qquad \frac{\partial P}{\partial y} = xe^{xy} + xe^{xy} + x^2ye^{xy} = 2xe^{xy} + x^2ye^{xy}$$

So $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \vec{F}$ is conservative.

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}, \qquad \frac{\partial f}{\partial y} = e^y + x^2e^{xy}$$

$$f = \int e^{xy}\,dx + \int xye^{xy}\,dx = \frac{1}{y}e^{xy} + \frac{xy}{y}e^{xy} - \int e^{xy}\,dx$$

$$= \cancel{\tfrac{1}{y}e^{xy}} + xe^{xy} - \cancel{\tfrac{1}{y}e^{xy}} + g(y) = xe^{xy} + g(y)$$

$$f = \int e^y\,dy + \int x^2e^{xy}\,dy = e^y + \frac{x^2}{x}e^{xy} + h(x) = e^y + xe^{xy} + h(x)$$

Thus, $\boxed{f(x,y) = e^y + xe^{xy} + K}$