Math 344: Calculus III
Chapter 13 Exam

Due: Tuesday, 14 May 2013

Late submissions will **NOT** be accepted!

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such (no rounded off decimals). Each problem is worth 10 points.

1. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle,$$

find its approximate location at time $t = 1.05$. (Decimal approximation is OK here, if necessary.)

$F(1,3) = \langle 1, -1 \rangle$

$\vec{r}(1.05) \approx \vec{r}(1) + .05 * \vec{F}(1)$

$= (1,3) + .05\,(1,-1)$

$= \boxed{(1.05, 2.95)}$

2. Find a potential function $f = f(x, y)$ for the conservative vector field

$$\mathbf{F}(x, y) = \langle y^2 \cos(x) + 3\cos(y), 2y\sin(x) - 3x\sin(y) \rangle.$$

$\frac{\partial f}{\partial x} = y^2 \cos x + 3 \cos y$

$\Rightarrow f = y^2 \sin x + 3x \cos y + g(y)$

$\frac{\partial f}{\partial y} = 2y \sin x - 3x \sin y$

$\Rightarrow f = y^2 \sin x + 3x \cos y + h(x)$

thus $\boxed{f(x,y) = y^2 \sin x + 3x \cos y + k}$

3. Evaluate the path integral $\int_C xe^y\,dx$, where C is the portion of the curve $y = \ln x$ from $(1,0)$ to $(e,1)$.

$$\int_C xe^y dx = \int_1^e x^2\,dx = \boxed{\frac{1}{3}(e^3-1)}$$

4. Find the work done by the force field

$$\mathbf{F}(x,y) = x\sin y\,\boldsymbol{i} + y\,\boldsymbol{j}$$

on a particle that moves along the parabola $y = x^2$ from $(-1,1)$ to $(2,4)$.

$$W = \int_C \vec{F}\cdot d\vec{r} \qquad \vec{r} = \langle x, x^2\rangle$$

$$dx = dx$$

$$dy = 2x\,dx$$

$$= \int_{-1}^{2} x\sin x^2\,dx + x^2\cdot 2x\,dx$$

$$= \int_{-1}^{2} 2x^3 + x\sin(x^2)\,dx \qquad u = x^2, \quad du = 2x\,dx$$

$$= \frac{1}{2}x^4\Big|_{-1}^{2} + \frac{1}{2}(-\cos x^2)\Big|_{-1}^{2}$$

$$= 8 - \frac{1}{2} + \left(-\frac{1}{2}\cos 4 + \frac{1}{2}\cos 1\right)$$

$$= \boxed{\frac{15}{2} + \frac{1}{2}\cos(1) - \frac{1}{2}\cos(4)}$$

5. Determine whether or not the vector field is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$. If not, give a reason why.

$$\mathbf{F}(x, y) = (x^3 + 4y)\,\boldsymbol{i} + (4xy - y^3)\,\boldsymbol{j}$$

$$\frac{\partial Q}{\partial x} = 4y \qquad \frac{\partial P}{\partial y} = 4$$

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y} \Rightarrow \boxed{\text{NOT conservative.}}$$

6. Find a function f such that $\mathbf{F} = \nabla f$, then use it to evaluate $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ along the path C.

$$\mathbf{F}(x, y) = \frac{y^2}{1 + x^2}\,\boldsymbol{i} + 2y \arctan x\,\boldsymbol{j},$$

$$C : \boldsymbol{r}(t) = t^2\,\boldsymbol{i} + 2t\,\boldsymbol{j}, \qquad 0 \leq t \leq 1.$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{1+x^2} \Rightarrow f = y^2 \arctan x + g(y)$$

$$\frac{\partial f}{\partial y} = 2y \arctan x \Rightarrow f = y^2 \arctan x + h(x)$$

$$\Rightarrow f(x,y) = y^2 \arctan x$$

$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$\vec{r}(1) = \langle 1, 2 \rangle$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = f(1,2) - f(0,0) = 4 \arctan(1) - 0$$

$$= 4\left(\pi/4\right) = \boxed{\pi}$$

7. Show that the path integral is independent of path, then evaluate the integral.

$$\int_C (1 - ye^{-x})\, dx + e^{-x}\, dy,$$

where C is any path between $(0, 1)$ and $(1, 2)$.

$$\left.\begin{array}{l} \frac{\partial Q}{\partial x} = -e^{-x} \\ \frac{\partial P}{\partial y} = -e^{-x} \end{array}\right\} \Rightarrow \vec{F} \text{ conservative}$$

$$\frac{\partial f}{\partial x} = 1 - ye^{-x} \Rightarrow f = x + ye^{-x} + g(y)$$

$$\frac{\partial f}{\partial y} = e^{-x} \Rightarrow f = ye^{-x} + h(x)$$

$$\Rightarrow f(x,y) = x + ye^{-x}$$

so $\int_C \vec{F} \cdot d\vec{r} = f(1,2) - f(0,1)$

$= 1 + 2e^{-1} - 1$

$= 2e^{-1}$

$= \boxed{\frac{2}{e}}$

8. Let $\mathbf{F}(x, y) = \left\langle y + e^{\sqrt{\arctan x}}, 2x - (\cos y^2)(\ln y) \right\rangle$ where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

By Green's Thm $\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D 1\, dA = A(D)$

$$\frac{\partial Q}{\partial x} = 2$$

$$\frac{\partial P}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$= \int_0^1 \sqrt{x} - x^2\, dx$

$= \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{3}$

$= \boxed{\frac{1}{3}}$

9. (*a.*) Show that if a vector field $\mathbf{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

F conservative $\Rightarrow$ $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

Then $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$ these are equal by Clairaut's Thm.

Similarly for the other two equalities.

(*b.*) Use this to show that the following path integral is *not* independent of path.

$$\int_C y\,dx + x\,dy + xyz\,dz$$

$\frac{\partial P}{\partial z} = 0$ but $\frac{\partial R}{\partial x} = yz$

$0 \neq yz$, so

this integral is <u>not</u> independent of path.

10. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with vector equation

$$r(t) = \langle 5\cos t - \cos 5t,\ 5\sin t - \sin 5t \rangle .$$

Use Green's theorem to find the area of the region that this curve encloses. [Hint: You might want to graph this curve first.]

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (5\cos t - \cos(5t))(5\cos t - 5\cos(5t)) - (5\sin t - \sin(5t))(-5\sin t + 5\sin(5t))\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 25\cos^2 t - 30\cos t\cos 5t + 5\cos^2(5t) + 25\sin^2 t - 30\sin t\sin(5t) + 5\sin^2(5t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 30\,dt - 15\underbrace{\int_0^{2\pi} \cos t\cos(5t) + \sin t\sin(5t)\,dt}$$

$= 0$ via WolframAlpha or very tedious trig substitutions.

$$= \boxed{30\pi}$$