Math 344: Calculus III
Chapter 12 Exam

Thursday, 18 April 2013

Name: KEY

Instructions: Complete 5 of the 6 problems, **showing all work**. You may pick one problem to be omitted or graded as a bonus. Clearly mark your choice; *e.g.,* circle the problem number and write "bonus." Each problem is worth 20 points, except the bonus which is worth 10 points.

1. Find the volume of the solid bounded above by the cone $z = \sqrt{x^2 + y^2}$, bounded below by the plane $z = 0$, and sitting above the disk $x^2 + y^2 \leq 81$.

$$V(S) = \iint_D z \, dA$$

$$= \int_0^{2\pi} \int_0^9 r \, (r \, dr \, d\theta) \quad \text{in polar coords}$$

$$= \int_0^{2\pi} \frac{1}{3} r^3 \Big|_0^9 \, d\theta$$

$$= 2\pi \cdot \frac{1}{3} \cdot 9^3$$

$$= 2\pi \, (243)$$

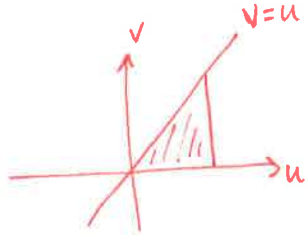
$$= \boxed{486\pi}$$

2. Evaluate the double integral

$$\int_0^1 \int_u^1 6\sqrt{1-v^2}\,dv\,du.$$

Need to change the limits

$$\left.\begin{array}{l} v:\ v=u \to v=1 \\ u:\ u=0 \to u=1 \end{array}\right\}$$



becomes

$v:\ 0 \to 1$

$u:\ 0 \to v$

the integral is now

$$\int_0^1 \int_0^v 6\sqrt{1-v^2}\,du\,dv$$

$$= \int_0^1 6u\sqrt{1-v^2}\Big|_0^v\,dv$$

$$= \int_0^1 6v\sqrt{1-v^2}\,dv \quad \longrightarrow$$

let $y = 1-v^2$

$dy = -2v\,dv$

$y(1) = 0$

$y(0) = 1$

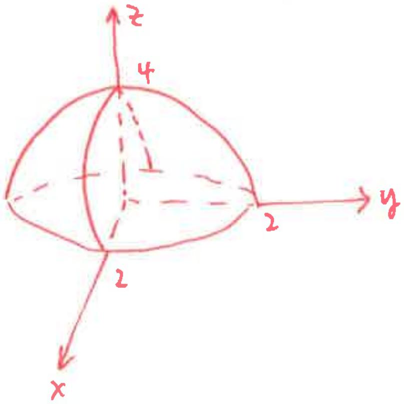
$$= \int_1^0 (-3)\sqrt{y}\,dy$$

$$= 3\int_0^1 \sqrt{y}\,dy = 3 \cdot \frac{2}{3}\, y^{3/2}\Big|_0^1 = \boxed{2}$$

3. Evaluate the triple integral

$$\iiint_E \sqrt{x^2+y^2}\,dV,$$

where E is the solid region bounded by the surface $z = 4 - x^2 - y^2$ and the xy-plane.



Smells like polar (i.e., cylindrical)

$z: 0 \longrightarrow 4-r^2$

$r: 0 \longrightarrow 2$

$\theta: 0 \longrightarrow 2\pi$

$\sqrt{x^2+y^2}$ becomes just r

dV becomes $r\,dz\,dr\,d\theta$

So $\displaystyle\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\cdot r\,dz\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi}\int_0^2 r^2 z\Big|_0^{4-r^2} dr\,d\theta$

$\displaystyle = \int_0^{2\pi}\int_0^2 4r^2 - r^4\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi} \frac{4}{3}r^3 - \frac{1}{5}r^5\Big|_0^2\,d\theta$

$= 2\pi\left(\frac{4}{3}(8) - \frac{1}{5}(32)\right)$

$= 2\pi\left(\frac{160-96}{15}\right)$

$= \boxed{\frac{148}{15}\pi}$

4. Use spherical coordinates and a triple integral to find the volume of the solid region bounded between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$. Check your answer *via* a formula from geometry.

$$V(E) = \iiint_E 1\, dV = \iiint_E \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$\rho : 2 \to 3$

$\theta : 0 \to 2\pi$

$\varphi : 0 \to \pi$

so $$V(E) = \int_0^{\pi}\int_0^{2\pi}\int_2^3 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}\rho^3 \sin\varphi \Big|_2^3 \, d\theta \, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}(27-8) \sin\varphi \, d\theta \, d\varphi$$

$$= \int_0^{\pi} 2\pi \cdot \frac{19}{3} \sin\varphi \, d\varphi$$

$$= \frac{38\pi}{3}\int_0^{\pi} \sin\varphi \, d\varphi$$

$$= \frac{38\pi}{3}(-\cos\varphi)\Big|_0^{\pi} = \frac{38}{3}\pi(1+1) = \boxed{\frac{76}{3}\pi}$$

From geometry $V(\text{sphere}) = \frac{4}{3}\pi r^3$

so $$V(E) = \frac{4}{3}\pi(3^3-2^3) = \frac{4}{3}\pi(27-8) = \frac{4}{3}\pi(19) = \boxed{\frac{76}{3}\pi}$$

:)

5. Make the given change of variables, then evaluate the integral.

$$\iint_D 4x^2 + 9y^2 \, dA, \quad x = 3u, \quad y = 2v,$$

where D is bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 4$.

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$, so $dA = 6\,du\,dv$ or $6\,dv\,du$

The region D becomes $\frac{(3u)^2}{9} + \frac{(2v)^2}{4} \leq 4$, or just $u^2+v^2 \leq 4$,

the disc of radius 2. This smells like polar coords.

$r: 0 \to 2$, $\theta: 0 \to 2\pi$, $du\,dv = r\,dr\,d\theta$, and

$$\iint_D 4x^2 + 9y^2 \, dA = \iint_S 4(3u^2) + 9(2v^2) \, (6\,du\,dv) = \iint_S (36u^2 + 36v^2)(6\,du\,dv)$$

$$= 216 \iint_S u^2 + v^2 \, du\,dv$$

$$= 216 \int_0^{2\pi} \int_0^2 r^2 \cdot r \, dr\,d\theta$$

$$= 216 \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^2 \, d\theta$$

$$= 216 \cdot 2\pi \cdot \frac{1}{4} \cdot 16$$

$$= \boxed{1728\pi}$$

6. If $m \le f(x,y) \le M$ for all $(x,y) \in D$, verify the inequality

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D),$$

where $A(D)$ is the area of the region D.

$$m \le f(x,y) \le M$$

$$\Rightarrow \iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$$

$$\Rightarrow m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$$

$$\Rightarrow m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D) \qquad \square$$

Use this inequality to estimate the value of $\iint_D \sin(x^2+y^2)\,dA$, where D is bounded by the curves $y = x^2$ and $y = 2 - x^2$.

$$-1 \le \sin(x^2+y^2) \le 1$$

$$x^2 = 2 - x^2$$

$$\Rightarrow \quad x: -1 \to 1$$

$$y: x^2 \to 2 - x^2$$

$$A(D) = \int_{-1}^{1}\int_{x^2}^{2-x^2} 1\,dy\,dx = \int_{-1}^{1} 2 - 2x^2\,dx$$

$$= 2\int_0^1 2 - 2x^2\,dx$$

$$= 2\left(2x - \tfrac{2}{3}x^3\right)\Big|_0^1$$

$$= 2\left(2 - \tfrac{2}{3}\right) = 2\left(\tfrac{4}{3}\right) = \boxed{\tfrac{8}{3}}$$

so $\boxed{-\tfrac{8}{3} \le \iint_D \sin(x^2+y^2)\,dA \le \tfrac{8}{3}}$