Math 344: Calculus III
Chapter 11 Exam

Due: Thursday, 28 March 2013

Late submissions will **NOT** be accepted!

Name: Key

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such (no rounded off decimals). Each problem is worth 10 points.

1. Consider the function $f(x, y, z) = xyz\, e^{\sqrt{x^2 - y^2}}$. (*a*) Find the domain of f, (*b*) evaluate $f(2, -2, 1)$.

$\text{domain}(f) = \{ (x,y,z) \mid x^2 - y^2 \geq 0 \}$

$f(2,-2,1) = 2(-2)(1)\, e^{\sqrt{0}} = \boxed{-4}$

2. A thin metal plate located in the xy-plane has temperature $T(x, y)$ at the point (x, y). The level curves of T are called *isothermals*. Identify the shape of the isothermals for $1 \leq k \leq 100$ for the function:

$$T(x, y) = \frac{100}{1 + x^2 + y^2}$$

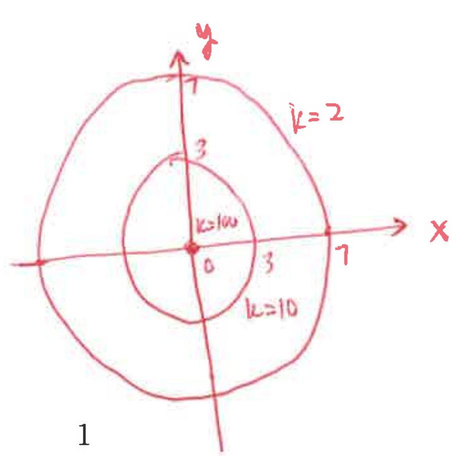
Sketch the isothermals for $T(x, y) = k = 2,\ 10,\ \text{and } 100$.

$$\frac{100}{1+x^2+y^2} = k \implies k + kx^2 + ky^2 = 100$$

$$\implies k(x^2 + y^2) = 100 - k$$

$$\implies x^2 + y^2 = \frac{100}{k} - 1$$

So the level curves are circles in the xy-plane.



3. Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

path 1: $x=0$: $\lim_{y\to 0} \frac{0}{|y|} = 0$

path 2: $y=x$: $\lim_{x\to 0} \frac{x^2}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}}$

$0 \neq \frac{1}{\sqrt{2}}$, so lim DNE.

4. Find the limit (you may assume it exists).

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Use polar coörds : $x^2+y^2=r^2$

limit becomes $\lim_{r\to 0^+} r^2 \ln(r^2) = 2 \lim_{r\to 0^+} r^2 \ln(r) = 0 \cdot (-\infty)$

Need L'Hôpital's rule: write $2 \lim_{r\to 0^+} \frac{\ln(r)}{1/r^2} \overset{L'H}{=} 2 \lim_{r\to 0^+} \frac{1/r}{-2/r^3}$

$= \lim_{r\to 0^+} -r^2 = \underline{\underline{0}}.$ $\square$

5. Use implicit differentiation (either version) to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$xz = \ln(y+z)$$

$$F(x,y,z) = xz - \ln(y+z)$$

$$\partial_x F = z$$

$$\partial_y F = -\frac{1}{y+z}$$

$$\partial_z F = x - \frac{1}{y+z} = \frac{xy + xz - 1}{y+z}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial_x F}{\partial_z F} = \frac{-z(y+z)}{xy+xz-1}}\ , \quad \boxed{\frac{\partial z}{\partial y} = \frac{-\partial_y F}{\partial_z F} = \frac{1}{xy+xz+1}}$$

6. Verify that the function $u(t,x) = e^{-\alpha^2 k^2 t}\sin(kx)$ is a solution of the *heat conduction equation*

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin(kx) = -\alpha^2 k^2 u$$

$$\frac{\partial u}{\partial x} = k e^{-\alpha^2 k^2 t} \cos(kx)$$

$$\frac{\partial^2 u}{\partial x^2} = -k^2 e^{-\alpha^2 k^2 t} \sin(kx) = -k^2 u$$

so $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = -\alpha^2 k^2 u$ also ✓

7. Let $z = x^2 + xy^3$, $x = uv^3 + w^2$, and $y = u + ve^w$. Find $\partial_u z$, $\partial_v z$, and $\partial_w z$.

$$\partial_u z = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \boxed{(2x+y^3)(v^3) + 3xy^2}$$

$$\partial_v z = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \boxed{(2x+y^3)(3uv^2) + 3xy^2 e^w}$$

$$\partial_w z = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = \boxed{(2x+y^3)(2w) + 3xy^2(ve^w)}$$

8. Prove the theorem: *Suppose f is a differentiable function of at least two variables. The maximum value of the directional derivative $D_{\boldsymbol{u}}f(\boldsymbol{x})$ is $\|\nabla f(\boldsymbol{x})\|$ and it occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.*

Pf| $D_{\vec{u}}f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = \|\nabla f(\vec{x})\| \, \|\vec{u}\| \cos\theta = \|\nabla f(\vec{x})\| \cos\theta$ since $\|\vec{u}\| = 1$.

This is maximum when $\theta = 0$ (so $\cos\theta = 1$). Thus its maximum value is $\|\nabla f(\vec{x})\|$, and $\theta = 0$ implies the $\vec{u}$ is in the same direction as $\nabla f(\vec{x})$. $\square$

9. Let $f(x, y, z) = \sqrt{x + yz}$, $P(3, 1, 1)$, and $v = \langle 6, 3, 2 \rangle$.

(*a*) Find $\nabla f(x, y, z)$,

(*b*) Find $\nabla f(P)$,

(*c*) Find $D_v f(P)$,

(*d*) Find a formula for the linearization $L(x, y, z)$ of f at P,

(*e*) Use part (*d*) to estimate the value of $f(3.01, 1.01, 0.99)$.

Clearly mark each part of your answer.

a) $\nabla f(x,y,z) = \boxed{\left\langle \frac{1}{2\sqrt{x+yz}}, \frac{z}{2\sqrt{x+yz}}, \frac{y}{2\sqrt{x+yz}} \right\rangle}$

b) $\nabla f(P) = \left\langle \frac{1}{2\sqrt{3+1}}, \frac{1}{2\sqrt{3+1}}, \frac{1}{2\sqrt{3+1}} \right\rangle = \boxed{\left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\rangle}$

c) $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 6,3,2 \rangle}{7}$ so $D_{\vec{v}} f(P) = D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$

$= \left\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\rangle \cdot \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle = \frac{6+3+2}{28} = \boxed{\frac{11}{28}}$

d) $L(x,y,z) = f(P) + D_x f(P)(x-3) + D_y f(P)(y-1) + D_z f(P)(z-1)$

$= \boxed{2 + \frac{1}{4}(x-3) + \frac{1}{4}(y-1) + \frac{1}{4}(z-1)}$

e) $L(3.01, 1.01, 0.99) = 2 + \frac{1}{4}\left(\frac{1}{100}\right) + \cancel{\frac{1}{4}\left(\frac{1}{100}\right)} - \cancel{\frac{1}{4}\left(\frac{1}{100}\right)}$

$= 2 + \frac{1}{400} = \boxed{2.0025}$

10. Find the maximum and minimum values of the function subject to the constraint.

$$f(x,y,z) = yz + xy; \quad \underbrace{xy = 1, \quad y^2 + z^2 = 1}$$

Lagrange Multiplier Method:

$\Rightarrow \quad xy = y^2 + z^2$

$\Rightarrow \quad g(x,y,z) = xy - y^2 - z^2 = 0$

$\nabla f(x,y,z) = \langle y, z+x, y \rangle$

$\nabla g(x,y,z) = \langle y, x-2y, -2z \rangle$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} y = \lambda y \\ z+x = \lambda(x-2y) \\ y = -2\lambda z \\ xy - y^2 - z^2 = 0 \end{cases}$$

$\Rightarrow \lambda = 1 \quad \dfrac{z+x}{x-2y} = \dfrac{y}{-2z}$

$z + x = x - 2y$

$y = -2z$

$-2z^2 - 2zx = xy - 2y^2$

$-2z^2 - 2zx = -2zx - 8z^2$

$6z^2 = 0$

$\Rightarrow z = 0 \Rightarrow y = 0$

$\Rightarrow x = \pm 1$

$\dfrac{z+x}{x-2y} = \dfrac{y}{-2z} = 1$

$-2z^2 - 2zx = xy - 2y^2$

$-2z^2 - 2zx = 1 - 2y^2$

$-2z^2 - 2zx = 1 - 2(1 - z^2)$

$-2z^2 - 2zx = -1 + 2z^2$

$1 - 2zx = 4z^2$

10.) Find the shortest distance from the point $(2,1,-1)$ to the plane $x+y-z=1. \Rightarrow z = x+y-1$

$$d^2(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$
$$= (x-2)^2 + (y-1)^2 + (z+1)^2$$

$d^2(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$ by inserting $z = x+y-1$

$$= x^2 - 4x + 4 + y^2 - 2y + 1 + x^2 + 2xy + y^2$$
$$= 2x^2 - 4x + 2y^2 - 2y + 2xy + 5$$

Call this map $F(x,y) = d^2(x,y)$.

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = \langle 4x - 4 + 2y;\ 4y - 2 + 2x \rangle$$

Max/Min happen when $\nabla F = \vec{0}$. So we need to solve

$$\begin{cases} 4x + 2y = 4 \\ 4y + 2x = 2 \end{cases} \Rightarrow \begin{cases} 4x + 2y = 4 \\ 4x + 8y = 4 \end{cases}$$

$$-6y = 0$$
$$y = 0 \Rightarrow x = 1$$
$$\Rightarrow z = 1 + 0 - 1 = 0$$

Then $d^2(1,0,0) = F(1,0) = 2 - 4 + 5 = 3$

so, shortest distance is

$$d(1,0,0) = \sqrt{d^2(1,0,0)} = \boxed{\sqrt{3}.}$$!!

FTIS: You should show it's a min (via S.D.T.), but I ran out of paper !!.