Math 344: Calculus III
Chapter 10 Exam

Thursday, 14 February 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such (no rounded off decimals). Each problem is worth 10 points.

1. (a) Write the definition of continuity at a point for a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

$\vec{r}$ is continuous at $t=a$ iff

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a).$$

(b) Show that the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ is continuous at $t = 0$.

$$\vec{r}(0) = \langle \cos 0, \sin 0, 0\rangle = \langle 1, 0, 0\rangle$$

$$\lim_{t\to 0} \vec{r}(t) = \left\langle \lim_{t\to 0} \cos t, \lim_{t\to 0} \sin t, \lim_{t\to 0} t \right\rangle$$

$$= \langle \cos 0, \sin 0, 0\rangle$$

$$= \langle 1, 0, 0\rangle \qquad \square$$

2. Find the curvature κ of the curve $y = x^4$ at the point $(1, 1)$.
[Hint: write the curve as a vector function.]

$t = 1.$

let $\begin{Bmatrix} x = t \\ y = t^4 \\ z = 0 \end{Bmatrix}$ so $\vec{r}(t) = \langle t, t^4, 0\rangle$

$$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3}$$

$\dot{\vec{r}}(t) = \langle 1, 4t^3, 0\rangle$, $\dot{\vec{r}}(1) = \langle 1, 4, 0\rangle$ $\Rightarrow$ $\|\dot{\vec{r}}(1)\|^3 = \left(\sqrt{1+16}\right)^3$

$= 17^{3/2}$

$\ddot{\vec{r}}(t) = \langle 0, 12t^2, 0\rangle$, $\ddot{\vec{r}}(1) = \langle 0, 12, 0\rangle$

$\dot{\vec{r}}(1) \times \ddot{\vec{r}}(1) = \langle 0, 0, 12\rangle$

$\|\dot{\vec{r}}(1) \times \ddot{\vec{r}}(1)\| = 12$

$$\Rightarrow \boxed{\kappa(1) = \frac{12}{17^{3/2}}}$$

3. (a) Find the domain of the vector function

$$\mathbf{r}(t) = \left\langle \arctan(7t), e^{-6t}, \frac{\ln(t)}{t} \right\rangle$$

$$\left.\begin{array}{l} \text{dom}(x) = \mathbb{R} \\ \text{dom}(y) = \mathbb{R} \\ \text{dom}(z) = (0,\infty) \end{array}\right\} \Rightarrow \boxed{\text{dom}(\vec{r}) = (0,\infty)}$$

or $0 < t < \infty$.

(b) Find $\lim_{t\to\infty} \mathbf{r}(t)$ for the same vector function.

$$\lim_{t\to\infty} \vec{r}(t) = \left\langle \lim_{t\to\infty} \arctan(7t), \lim_{t\to\infty} e^{-6t}, \lim_{t\to\infty} \frac{\ln t}{t} \right\rangle$$

$$= \boxed{\left\langle \pi/2, 0, 0 \right\rangle}$$

4. Find $\dot{\mathbf{r}}(t)$ for $\mathbf{r}(t) = e^{t^2}\mathbf{i} + \mathbf{j} + \ln(1+3t)\mathbf{k}$.

$$\dot{\vec{r}}(t) = \left\langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \right\rangle$$

$$= \boxed{\left\langle 2te^{t^2}, 0, \frac{3}{1+3t} \right\rangle}$$

5. Find the velocity $\mathbf{v}$, speed v, and acceleration $\mathbf{a}$ of the particle.

$$\mathbf{r}(t) = \langle t\sin t, t\cos t, 7t^2 \rangle$$

$$\vec{v} = \dot{\vec{r}} = \langle \sin t + t\cos t,\ \cos t - t\sin t,\ 14t \rangle$$

$$v = \|\vec{v}\| = \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + 196t^2}$$

$$= \sqrt{\underline{\sin^2 t} + \cancel{2t\sin t\cos t} + \underline{\underline{t^2\cos^2 t}} + \underline{\cos^2 t} - \cancel{2t\sin t\cos t} + \underline{\underline{t^2\sin^2 t}} + 196t^2}$$

$$= \sqrt{1 + t^2 + 196t^2}$$

$$= \sqrt{1 + 197t^2}$$

$$\vec{a} = \dot{\vec{v}} = \langle \cos t + \cos t - t\sin t,\ -\sin t - \sin t - t\cos t,\ 14 \rangle$$

$$= \langle 2\cos t - t\sin t,\ -2\sin t - t\cos t,\ 14 \rangle$$

6. Reduce the equation to one of the standard forms, and classify the quadratic surface.

$$4x^2 + y^2 - 4z^2 - 4y + 40z + 100 = 0$$

Sketch the surface for extra credit (+2 points).

$$4x^2 + y^2 - 4y + 4 - 4(z^2 - 10 + 25) = -100 + 4 - 100$$

$$4x^2 + (y-2)^2 - 4(z-5)^2 = -196$$

$$\frac{-4}{196}x^2 - \frac{(y-2)^2}{196} + \frac{4}{196}(z-5)^2 = 1$$

$\Rightarrow$ $\boxed{\text{Hyperboloid of two sheets.}}$

7. Show that $\mathbf{T} \perp \dot{\mathbf{T}}$.

Recall: $\|\vec{T}\|^2 = \vec{T}\cdot\vec{T} = 1$

take the derivative:

$$\frac{d}{dt}\left[\vec{T}\cdot\vec{T} = 1\right]$$

$$\Rightarrow \quad \dot{\vec{T}}\cdot\vec{T} + \vec{T}\cdot\dot{\vec{T}} = 0$$

$$\Rightarrow \quad 2\dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \quad \dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \quad \dot{\vec{T}} \perp \vec{T} \quad \square$$

8. Find the arc length of the curve $\mathbf{r}(t) = \langle 2\sin t, 3t, 2\cos t\rangle$ for $-4 \leq t \leq 4$.

$$s = \int_{-4}^{4} \|\dot{\vec{r}}(t)\|\, dt = \int_{-4}^{4} \sqrt{4\cos^2 t + 9 + 4\sin^2 t}\, dt$$

$$= \int_{-4}^{4} \sqrt{13}\, dt$$

$$= 2\int_{0}^{4} \sqrt{13}\, dt$$

$$= \boxed{8\sqrt{13}}$$

9. Show that the curvature of a circle of radius a is equal to $1/a$.

$$\kappa = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$$

$$\vec{r} = \langle a\cos t, a\sin t\rangle$$

$$\dot{\vec{r}} = \langle -a\sin t, a\cos t\rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = |a| = a \quad (\text{since } a = \text{radius} > 0)$$

$$\vec{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} = \frac{1}{a}\langle -a\sin t, a\cos t\rangle = \langle -\sin t, \cos t\rangle$$

$$\dot{\vec{T}} = \langle -\cos t, -\sin t\rangle$$

$$\|\dot{\vec{T}}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\Rightarrow \kappa = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \boxed{\frac{1}{a}}$$

□

10. Find **T**, **N**, **B**, and κ for the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ at the point $(1, 0, 0)$.

$$\vec{r} = \langle \cos t, \sin t, t \rangle$$

$$\dot{\vec{r}} = \langle -\sin t, \cos t, 1 \rangle \qquad \|\dot{\vec{r}}\| = \sqrt{2}$$

$$\vec{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$\dot{\vec{T}} = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \qquad \|\dot{\vec{T}}\| = \frac{1}{\sqrt{2}}$$

$$\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|} = \langle -\cos t, -\sin t, 0 \rangle$$

So $\vec{T}(0) = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle = \boxed{\langle 0, 1/\sqrt{2}, 1/\sqrt{2} \rangle}$

$\vec{N}(0) = \boxed{\langle -1, 0, 0 \rangle}$

Then $\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \boxed{\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$

$$\kappa(0) = \frac{\|\dot{\vec{T}}(0)\|}{\|\dot{\vec{r}}(0)\|} = \frac{\frac{1}{\sqrt{2}}}{(\sqrt{2})} = \frac{1}{(\sqrt{2})^2} = \boxed{\frac{1}{2}}$$

Bonus. Suppose that a particle curve has [redacted] acceleration $\mathbf{a} = \mathbf{N}$. The initial position of the particle is $\mathbf{r}(0) = \langle 1, 0, 0\rangle$, and the initial velocity is $\mathbf{v}(0) = \langle 0, 1, 7\rangle$. If $\mathbf{T} = \frac{1}{\sqrt{50}}\langle -\sin t, \cos t, 7\rangle$, find the position function $\mathbf{r}(t)$ of the particle.

$$\vec{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} = \frac{1}{\sqrt{50}}\langle -\sin t, \cos t, 7\rangle$$

$$\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|} = \frac{\frac{1}{\sqrt{50}}\langle -\cos t, -\sin t, 0\rangle}{1/\sqrt{50}} = \langle -\cos t, -\sin t, 0\rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle -\sin t + C_1, \cos t + C_2, C_3\rangle$$

$$\vec{v}(0) = \langle 0, 1, 7\rangle = \langle 0 + C_1, 1 + C_2, 0 + C_3\rangle$$

$$\Rightarrow \quad \vec{v}(t) = \langle -\sin t, \cos t, 7\rangle$$

$$\vec{r} = \int \vec{v}(t)\,dt = \langle \cos t + C_1, \sin t + C_2, 7t + C_3\rangle$$

$$\vec{r}(0) = \langle 1, 0, 0\rangle = \langle 1 + C_1, 0 + C_2, 0 + C_3\rangle$$

so $\boxed{\vec{r}(t) = \langle \cos t, \sin t, 7t\rangle}$