Name: Key
M243: Calculus II (Sp 2019)
Quiz 10 (In class)

Instructions. Complete all exercises, showing enough work. You may work in groups, but be sure to write your own solutions.

True–False *If the statement is always true neatly write* **T** *on the line. Otherwise, write* **F**. *If true, explain why. If false, give a counter-example.*

F **1.** If the parametric curve $x = x(t)$, $y = y(t)$ satisfies $\dot{y}(1) = 0$, then it has a horizontal tangent line when $t = 1$.

$\dot{x}(1)$ could be 0 or undefined.

F **2.** If $x = x(t)$ and $y = y(t)$ are twice differentiable, then $\dfrac{d^2y}{dx^2} = \dfrac{\ddot{y}(t)}{\ddot{x}(t)}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(dy/dx)^{\bullet}}{\dot{x}} = \frac{(\dot{y}/\dot{x})^{\bullet}}{\dot{x}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}$$

T **3.** The length of the parametrized curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ from $t = a$ to $t = b$ is $\displaystyle\int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(\dot{x}\,dt)^2 + (\dot{y}\,dt)^2} = \sqrt{(\dot{x}^2 + \dot{y}^2)\,dt^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$$

F **4.** If a point is represented by (x, y) in Cartesian coordinates with $x \neq 0$ and (r, θ) in polar coordinates, then $\theta = \arctan(\frac{y}{x})$.

This is only true in Quadrant I. In QII and QIII: $\theta = \pi + \arctan(y/x)$
In QIV: $\theta = 2\pi + \arctan(y/x)$

T **5.** The polar curves $r = 1 - \sin(2\theta)$ and $r = \sin(2\theta) - 1$ have the same graph.

They trace the same curve but in opposite orientations.

T **6.** The equations $r = 2$, $x^2 + y^2 = 4$, and $\mathbf{r}(t) = \langle 2\sin(3t), 2\cos(3t)\rangle$ $(0 \leq t \leq 2\pi)$ all have the same graph.

They are all circles centered at the origin w/ radius 2.

F **7.** The parametric equations $\mathbf{r}_1(t) = \langle t^2, t^4\rangle$ and $\mathbf{r}_2(t) = \langle t^3, t^6\rangle$ have the same graphs.

For $\vec{r}_1$, range$(x) = [0, \infty)$ but for $\vec{r}_2$, range$(x) = \mathbb{R}$.

8. Find polar equations, $r = r(\theta)$, for the curves

a.) $x + y = 2$

$r\cos\theta + r\sin\theta = 2$

$r(\cos\theta + \sin\theta) = 2$

$$\boxed{r = \frac{2}{\cos\theta + \sin\theta}}$$

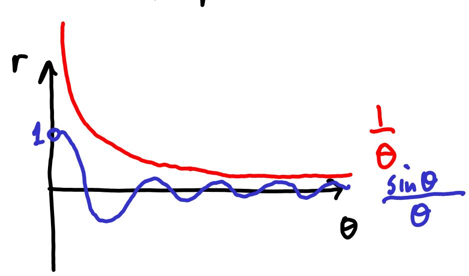
b.) $x^2 + y^2 = 2$

$r^2 = 2$

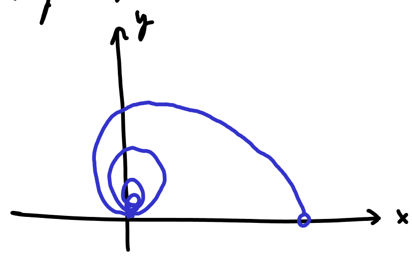
$$\boxed{r = \sqrt{2}}$$

9. The curve with polar equation $r = \sin(\theta)/\theta$ is called a **cochleoid**. Try to sketch the cochleoid by hand, then graph it using a CAS and compare.

In the (r,θ)-plane, $r = \dfrac{\sin\theta}{\theta}$ has a graph that looks like:

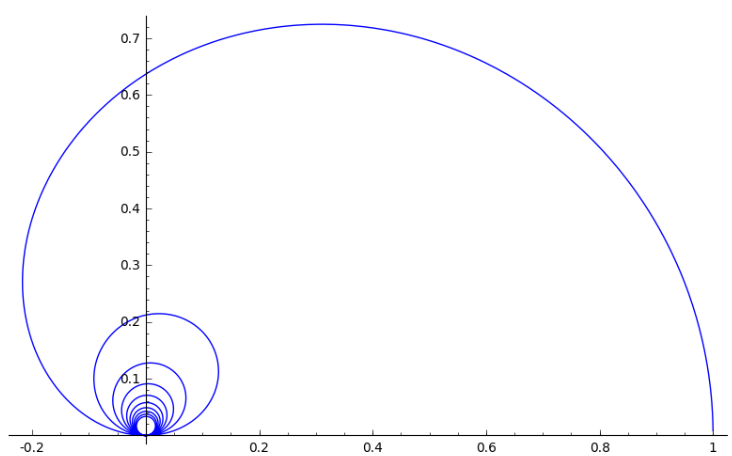


So the cochleoid might look something like:



Done w/ SageMath:

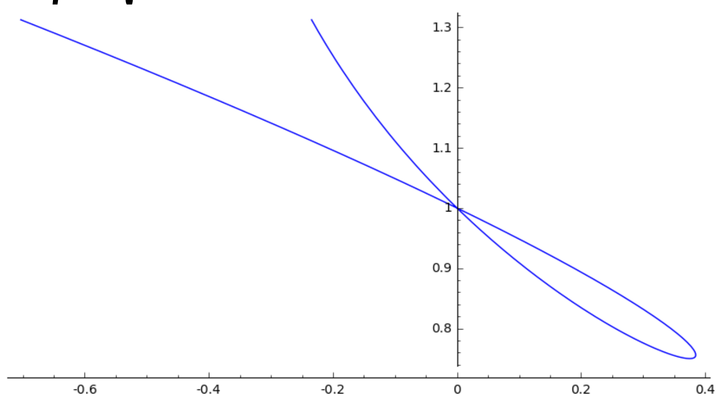
```
th = var('th',latex_name=r'\theta');
r = sin(th)/th;
parametric_plot([r*cos(th),r*sin(th)],(th,0,5*2*pi),plot_points=4000)
```



10. Find the area enclosed by the loop of the parametric curve

$$\begin{cases} x = t^3 - t, \\ y = t^2 + t + 1 \end{cases}$$

start by looking at the graph. Again, using SageMath:



This is plotted for t-values from -1.25 to 0.25.

By the graph, put $y=1$, and solve for t:

$1 = t^2+t+1 \rightarrow t(t+1)=0$

$\Rightarrow t=0,\ t=-1$.

$x(0) = 0^3 - 0 = 0$

$x(-1) = (-1)^3 - (-1) = -1+1 = 0$

so these points correspond to the point where the curve self-intersects.

$t = -1, 0$

$$A = \int_{t_0}^{t_1} x\,dy = \int_{-1}^{0} (t^3-t)(2t+1)\,dt = \int_{-1}^{0} 2t^4+t^3-2t^2-t\,dt$$

$$= \frac{2}{5}t^5 + \frac{1}{4}t^4 - \frac{2}{3}t^3 - \frac{1}{2}t^2 \Big|_{-1}^{0}$$

$$= -\left(-\frac{2}{5}+\frac{1}{4}+\frac{2}{3}-\frac{1}{2}\right)$$

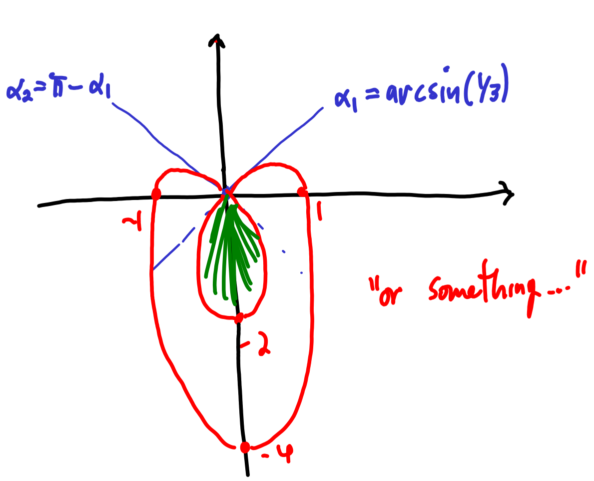
$$= -\left(\frac{-9}{10}+\frac{11}{12}\right)$$

$$= -\left(\frac{110-108}{120}\right) = -\frac{1}{60}$$

but area should be positive, so I must have integrated in the wrong direction.

$$\boxed{\text{Area} = \frac{1}{60}}$$

11. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.



$r = 0: \quad \sin\theta = 1/3$

$\theta = \arcsin(1/3)$

$$A = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$= \frac{1}{2}\int_{\alpha_1}^{\pi-\alpha_1} (1-3\sin\theta)^2\,d\theta$$

$$= \int_{\alpha_1}^{\pi/2} 1 - 6\sin\theta + 9\sin^2\theta\,d\theta$$

$$= \int_{\alpha_1}^{\pi/2} \frac{11}{2} - 6\sin\theta - \frac{9}{2}\cos(2\theta)\,d\theta$$

$$= \frac{11}{2}\theta + 6\cos\theta - \frac{9}{4}\sin(2\theta)\Big|_{\alpha_1}^{\pi/2}$$

$: \cos(\alpha_1) = \frac{\sqrt{8}}{3}$

$$= \frac{11}{2}\left(\frac{\pi}{2} - \alpha_1\right) + 6\cos\frac{\pi}{2} - 6\cos\alpha_1 - \frac{9}{2}\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{9}{2}\cos\alpha_1\sin\alpha_1$$

$$= \frac{11\pi}{4} - \frac{11\sqrt{8}}{6} - 2\sqrt{8} + \frac{9}{2}\,\frac{\sqrt{8}\cdot 1}{9}$$

$$= \frac{11\pi}{4} - \frac{11\sqrt{2}}{3} - 4\sqrt{2} + \sqrt{2} = \frac{11\pi}{4} - \frac{20}{3}\sqrt{2} = \boxed{\frac{33\pi - 80\sqrt{2}}{12}}$$

12. Find the length of the parametric curve segment,

$$\begin{cases} x = 2 + 3t, \\ y = \cosh(3t), \\ 0 \le t \le 1. \end{cases}$$

$\dot{x} = 3$ $\quad \dot{x}^2 = 9$

$\dot{y} = 3\sinh(3t)$ $\quad \dot{y}^2 = 9\sinh^2(3t)$

$ds = 3\sqrt{1 + \sinh^2(3t)}\ dt$

Hyperbolic Pythagorean Theorem: $\cosh^2 x - \sinh^2 x = 1$

so $ds = 3\sqrt{\cosh^2(3t)}\, dt$ but cosh is always ≥ 1.

so $ds = 3\cosh(3t)\,dt$ and

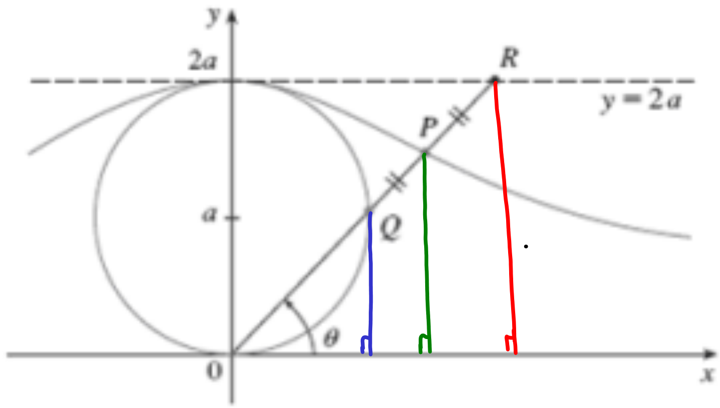
$$s = \int_0^1 3\cosh(3t)\,dt = \sinh(3t)\Big|_0^1$$

$$= \sinh(3) - \sinh(0)$$

$$= \boxed{\sinh(3)}$$

$$= \boxed{\frac{1}{2}e^3 - \frac{1}{2}e^{-3}}$$

13. In the figure the circle of radius a is stationary, and for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.



As a polar curve the circle is given by $r = 2a\sin\theta$.

So the coord's of Q are: $\begin{cases} x = 2a\sin\theta\cos\theta \\ y = 2a\sin\theta\sin\theta \end{cases}$

R lies on the line $\overline{OQ}$ w/ y-value $2a$:

$$OQ: \quad y = \frac{\cancel{2a}\ \cancel{\sin\theta}\sin\theta}{\cancel{2a}\ \cancel{\sin\theta}\cos\theta}\cdot x \quad , \text{ so}$$

$$R: \quad \begin{cases} x = 2a\cot\theta \\ y = 2a \end{cases}$$

P is then the midpoint of Q and R:

$$P: \quad \begin{cases} x = \frac{1}{2}(2a\sin\theta\cos\theta + 2a\cot\theta) \\ y = \frac{1}{2}(2a\sin^2\theta + 2a) \end{cases}$$

Simplifying, the longbow is parametrized by:

$$\boxed{\begin{aligned} x(\theta) &= a(\sin^2\theta + 1)\cot\theta \\ y(\theta) &= a(\sin^2\theta + 1) \end{aligned}}$$