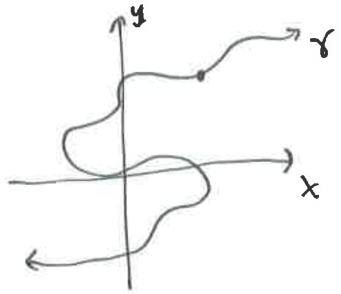
9 : Parametric Equations and Polar coords

9.1 : Parametric Curves



γ is not a function, but at any point (x,y) we can think of

$$(x(t), y(t)) = \gamma(t)$$

as the position of γ at some time t, so x,y are functions of a parameter t.

the equns $\left\{ \begin{array}{l} x = f(t) = x(t) \\ y = g(t) = y(t) \end{array} \right\}$ are called parametric eqn's.

t does not necessarily represent time, and we could use any other letter, but t is "usually" time in applications.

Ex. Sketch: $x = t^2 - 2t$
$y = t + 1$

We could make a table of values, and plot points,

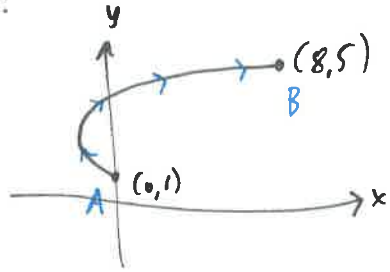
also $x = (y-1)^2 - 2(y-1)$
$= y^2 - 2y + 1 - 2y + 2$
$= y^2 - 4y + 3$

So the parametric curve $\gamma = (x(t), y(t))$ is also given by $y^2 - 4y + 3$.

Since t can be anything in this example, we get the whole parabola.

Ex. What if we restrict to $x = t^2 - 2t$
$y = t + 1$
$0 \le t \le 4$

Then we get:



If we think of r as a particle, then it "moves from A to B."

here $r(0) = (x(0), y(0)) = (0,1)$ is the initial pt and
$r(4) = (x(4), y(4)) = (8,5)$ is the terminal pt.

In general, if $r(t) = (x(t), y(t))$, $a \le t \le b$, then
$r(a)$ is initial and $r(b)$ is terminal.

Ex. $\left.\begin{aligned} x &= \cos t \\ y &= \sin t \\ 0 &\le t \le 2\pi \end{aligned}\right\}$ $x^2 + y^2 = 1 \Rightarrow$ Unit circle!

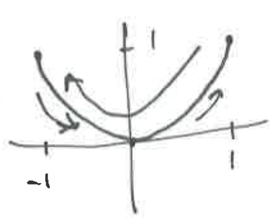
Ex. $\left.\begin{aligned} x &= \sin 2t \\ y &= \cos 2t \\ 0 &\le t \le 2\pi \end{aligned}\right\}$ travels the circle twice! (starting at "$\frac{\pi}{2}$")

Ex. Find parametric eqns for a circle centered at (h,k) w/ radius r.

$$x = h + r\cos t$$
$$y = k + r\sin t$$
$$t \in [0, 2\pi]$$

Ex. Sketch $r(t) = (x(t), y(t))$ if $x = \sin t$ and $y = \sin^2 t$.

Notice $y = x^2$ but as t varies $-1 \le x, y \le 1$

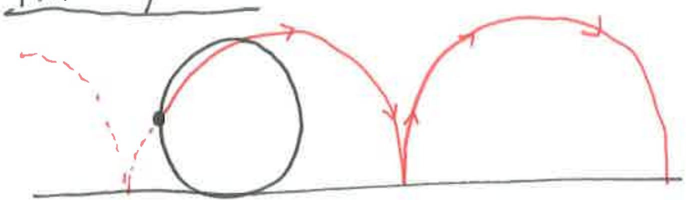


On a calculator or wolfram:

$$\left.\begin{aligned} x &= t + 2\sin 2t \\ y &= t + 2\cos 5t \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= 1.5\cos t - \cos 30t \\ y &= 1.5\sin t - \sin 30t \end{aligned}\right\}, \qquad \text{and}$$

$$\left.\begin{aligned} x &= \sin(t + \cos 100t) \\ y &= \cos(t + \sin 100t) \end{aligned}\right\}$$

The Cycloid



more "flat"

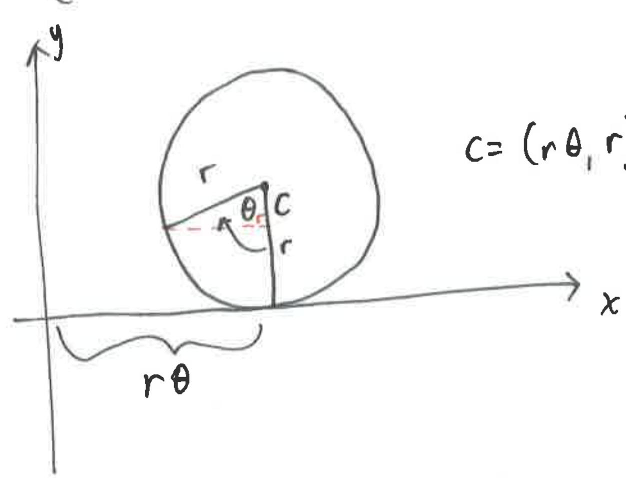
Pick a point on a "wheel" and roll the wheel. The curve that the point traces is called a cycloid.

Exercise for students:

Work through Example 7 on pg. 485 to derive formula $\boxed{1}$:

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \\ \theta \in \mathbb{R} \end{cases} \qquad r(r,\theta) = (x(r,\theta), y(r,\theta))$$

where



$C = (r\theta, r)$

Picture makes sense for "first rotation", but actually holds for all θ.

Christian Huygens: tautochrone problem:

No matter where a particle P is placed on an inverted cycloid, it takes the same speed to fall to the bottom.

time is the same!

Used to keep time in pendulum clocks (doesn't matter how far you swing it; always takes a "second").

Ex. (5) $x = 3t - 5$
$y = 2t + 1$

Ex (10) $x = 4\cos\theta$
$y = 5\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Ex. (9) $x = \sin\theta$
$y = \cos\theta$ $\quad 0 \le \theta \le \pi$

40. Swallow tail catastrophe curves

$$x = 2ct - 4t^3$$
$$y = -ct^2 + 3t^4$$

41. Lissajous figures

$$x = a \sin nt \qquad n \in \mathbb{Z}^+$$
$$y = b \cos t$$

9.2: Calculus w/ parametrized curves

We can still define tangent lines on these curves with no ambiguity.

then, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad , \text{ or}$$

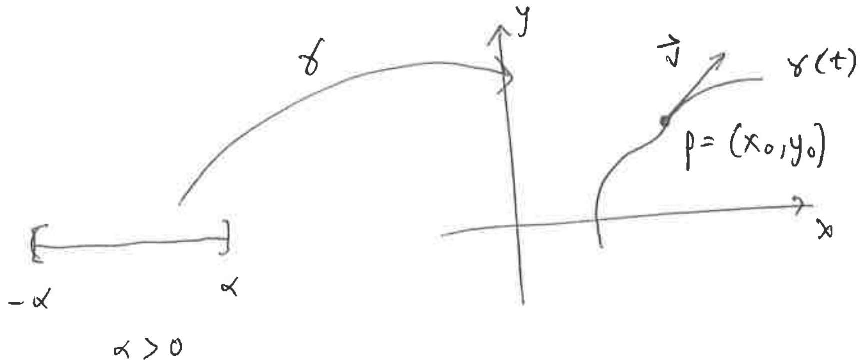
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if } dx/dt \neq 0.$$

<u>Exc.</u> Find $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

Hint: it's not $\dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$!

§9.2 cont'd:

Let $\gamma(t) = (x(t), y(t))$ be a parametrized curve in $\mathbb{R}^2$, and the p be a point in $\mathbb{R}^2$ that γ passes through; i.e.,



$\gamma(0) = p$

↑ position of particle

velocity of particle

$\vec{v}$ is a tangent vector to γ at p: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) = \vec{v}$

The direction of $\vec{v}$ is the slope of the tangent line to γ at p:

$$\left.\frac{dy}{dx}\right|_p = \boxed{\cancel{\dot{\gamma}(0) = \frac{d\gamma}{dt}(0)}}$$

$$\boxed{\cancel{\dot{\gamma} = \frac{d\gamma}{dt}}} = \left.\frac{dy}{dt}\cdot\frac{dt}{dx}\right|_{t=0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=0} \text{ when } \frac{dx}{dt} \neq 0.$$

In general, at points along γ, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Second derivative is not so simple:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{dx/dt}$$ ← need quotient rule!

$$= \left(\frac{dx}{dt}\right)^{-1}\cdot\frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{(dx/dt)^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

In Newtonian notation: $$\boxed{\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}}$$

* We usually don't need to actually use this formula b/c we already know a formula for dy/dx in terms of t. But it's a good exercise. ☺

Ex. let $\gamma(t) = (t^2, t^3 - 3t)$

so $x(t) = t^2$
$y(t) = t^3 - 3t$

Find dy/dx and d^2y/dx^2, and find the points on γ where the tangent is horizontal or vertical and the intervals of concavity.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{3t^2-3}{2t}}$$

Horizontal when $dy/dt = 0$: $3t^2 - 3 = 0$

$3(t^2-1) = 0$

$t = \pm 1$

$\gamma(1) = (1,-2)$, $\gamma(-1) = (1, 2)$

Vertical when $dx/dt = 0$: $2t = 0$

$t = 0$

$\gamma(0) = (0,0)$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{1}{2t} \cdot \frac{2t(6t) - (3t^2-3)(2)}{(2t)^2} =$$

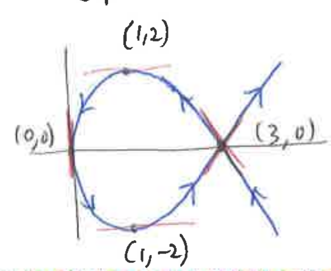
$$= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6}{8} \cdot \frac{t^2+1}{t^3} = \boxed{\frac{3}{4} \cdot \frac{t^2+1}{t^3}}$$

t^2+1 is always positive

t^3 is $\begin{cases} \text{neg for } t<0 \\ \text{pos for } t>0 \end{cases}$

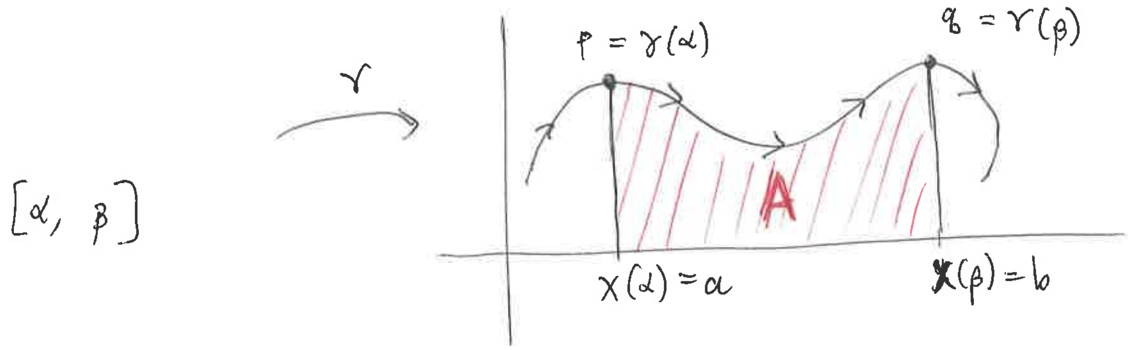
so $\gamma(t)$ is concave up on $(0,\infty)$ and concave down on $(-\infty, 0)$.

graph:



Areas:

Again, let $\gamma(t) = (x(t), y(t))$ be a parametrized curve. Now assume that the parametrization is such that the curve is only traversed once as t increases, and that the "particle" moves in the positive direction. i.e.,



We know that

$$A = \int_a^b y \, dx$$

if we can write γ as a function $y = f(x)$.

By substituting, we have

$$y = y(t) \quad \text{and} \quad dx = \frac{dx}{dt} dt = \dot{x}(t)\, dt, \quad \text{so}$$

$$\boxed{A = \int_\alpha^\beta y(t)\, \dot{x}(t)\, dt}$$

Ex. Find the area under one "cycle" of the cycloid

$$\gamma(\theta) = (r(\theta - \sin\theta), r(1-\cos\theta))$$

$$y(\theta) = r(1-\cos\theta) \qquad \dot{x}(\theta) = r(1-\cos\theta)\, d\theta$$

$\frac{1}{2}(1+\cos 2\theta)$

So, $$A = \int_0^{2\pi} r^2 (1-\cos\theta)^2 \, d\theta = r^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$= r^2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = r^2(2\pi + \pi - 0) = \boxed{3\pi r^2}!$$

Arc Length:

Again, we already know how to find the length of a curve between 2 points when the curve can be written as $y = f(x)$:

$$L(a,b) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If we can paramatrize a curve γ by $\gamma(t) = (x(t), y(t))$ such that $t \in [\alpha, \beta]$ and $\dot{x}(t) = \frac{dx}{dt} > 0$ for all t (basically the same condition as for area), then

$$L\gamma(\alpha,\beta) = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

But since $dx/dt > 0$, we can move it in the $\sqrt{\cdot}$ to get:

$$L\gamma(\alpha,\beta) = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt \qquad (*)$$

Notice that this is consistent w/ the formula:

$$L = \int ds \quad \text{where} \quad ds^2 = dx^2 + dy^2$$

It turns out eq'n $(*)$ works for any parametrization as above. (gives the same answer.)

<u>Ex.</u> Use this method to find circumference of a circle:

$r(\theta) = (r\cos\theta, r\sin\theta) \qquad \theta \in [0, 2\pi]$

$$L\gamma(0,2\pi) = \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta = r\theta\Big|_0^{2\pi} = \boxed{2\pi r}.$$

Ex. Find the length of one cycle of cycloid:

$$r(\theta) = \left(r(\theta - \sin\theta),\ r(1-\cos\theta)\right), \quad 0 \le \theta \le 2\pi$$

$$L_r(0, 2\pi) = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi} r\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\ d\theta$$

$$= \sqrt{2}\, r \int_0^{2\pi} \sqrt{1 - \cos\theta}\ d\theta$$

$$\tfrac{1}{2}(1-\cos 2\theta) = \sin^2\theta \Rightarrow 1 - \cos\theta = 2\sin^2\left(\tfrac{\theta}{2}\right)$$

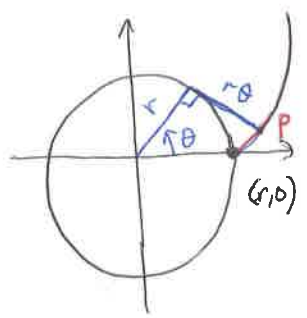
$$= \sqrt{2}\, r \int_0^{2\pi} \sqrt{2\sin^2\left(\tfrac{\theta}{2}\right)}\ d\theta$$

$$= 2r \int_0^{2\pi} \sin\left(\tfrac{\theta}{2}\right) d\theta$$

$$= -2r \cdot 2\cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi} = -4r\cos\pi + 4r\cos 0$$

$$= 4r + 4r = \boxed{8r}\ !$$

Ex. 53

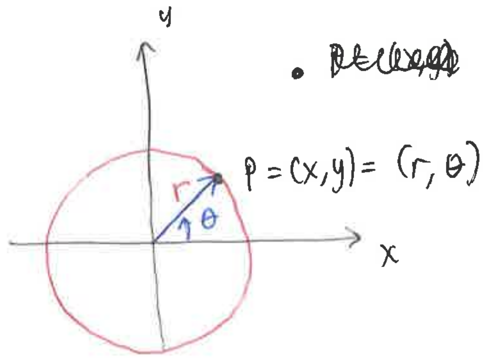


← the involute of the circle.

Show that $r(t) = \left(r(\cos\theta + \theta\sin\theta),\ r(\sin\theta - \theta\cos\theta)\right)$

* This can be an X.C. project for this section. Show all work.

§9.3: Polar Coordinates

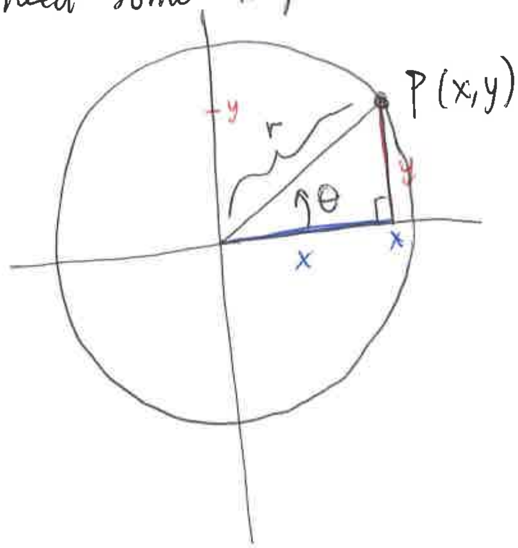


Any pt in $\mathbb{R}^2$ can be described by the radius of the circle that it lies on, and the angle that its "position vector" makes w/ x^+-axis.

$$P=(r,\theta).$$

P has (x,y)- coords and (r,θ)- coords. We should be able to write one in terms of the other (i.e, swith between them).

To do so, we need some trig:



Pythagorean Thm : $r^2 = x^2+y^2 \Rightarrow \boxed{r=\sqrt{x^2+y^2}} \geq 0$

$$r=0 \Leftrightarrow P=(0,0)$$

Right Δ trig: $\tan\theta = \frac{O}{A} = \frac{y}{x} \Rightarrow \boxed{\theta = \arctan\left(\frac{y}{x}\right)}$

These formulae turn (x,y) into (r,θ) (or Cartesian $\rightarrow$ Polar)

$$P(x,y)_c = \left(\sqrt{x^2+y^2},\ \arctan\left(\tfrac{y}{x}\right)\right)_p$$

Also by Right Δ trig:

$$\cos\theta = \frac{x}{r}, \quad \text{so} \quad \boxed{x = r\cos\theta}$$

$$\text{and } \sin\theta = \frac{y}{r}, \quad \text{so} \quad \boxed{y = r\sin\theta}$$

(*) These formulae change Polar to Cartesian:

$$C(r,\theta)_p = (r\cos\theta, r\sin\theta)_c$$

$$P(C(r,\theta)_p) = (r,\theta)_p, \text{ and}$$
$$C(P(x,y)_c) = (x,y)_c$$

so P and C are inverse transformations.

Ex. Convert $(2, \pi/3)$ from polar to Cartesian.

$$x = r\cos\theta = 2\cos(\pi/3) = 2(\tfrac{1}{2}) = 1$$
$$y = r\sin\theta = 2\sin(\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$$

$$\text{so } C(2,\pi/3)_p = \boxed{(1,\sqrt{3})_c}$$

Ex. Convert $(-1,1)$ from Cartesian to polar:

$$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{1}{-1}\right) = -\pi/4$$

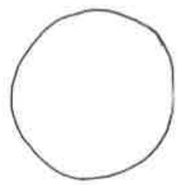
$$\text{so, } P(-1,1)_c = \boxed{(\sqrt{2}, -\pi/4)_p}$$

Polar Curves.

the graph of a polar equation $r = f(\theta)$ (or $F(r,\theta) = 0$) (r,θ)

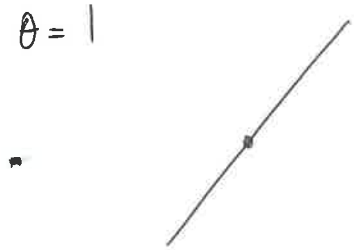
consists of all points who have at least one polar rep.$^{\vee}$ that satisfies the eqn.

Ex. $r = 2$



circle w/ radius 2.

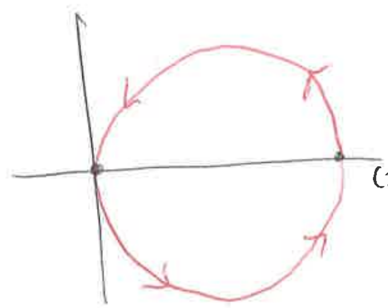
Ex. $\theta = 1$



line w/ angle $\theta = 1$ (slope $= \tan 1$)

Ex. $r = 2\cos\theta$

θ	$r = 2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	-2



circle w/ center $(1,0)$ and $r = 1$.

Write in Cartesian coords:

$$r = 2\frac{x}{r}$$

$$r^2 = 2x$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

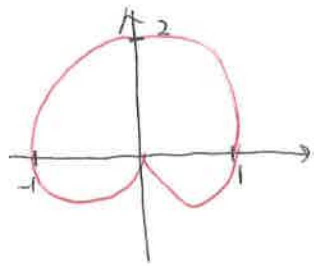
Ex. The Cardiod.

$$r = 1 + \sin\theta$$

$\sin\theta$: $0 \to \frac{\pi}{2} \to \pi \to \frac{3\pi}{2} \to 2\pi$

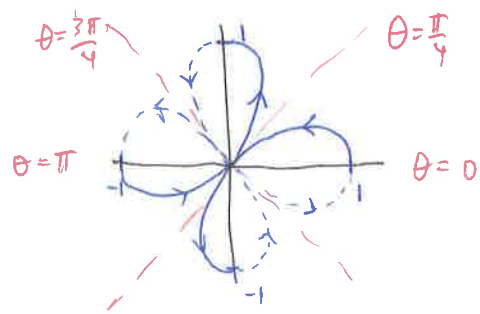
$0 \xrightarrow{inc} 1 \xrightarrow{dec} 0 \xrightarrow{dec} -1 \xrightarrow{inc} 0$

$1+\sin\theta$: $1 \xrightarrow{inc} 2 \xrightarrow{dec} 1 \xrightarrow{dec} 0 \xrightarrow{inc}$



← horrible picture. Use computer?

Ex. Four-leaved rose $r = \cos 2\theta$



Derivatives: tangents to polar curves.

Idea: regard θ as a parameter, $r = f(\theta)$, and write

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

then $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{dy/d\theta}{dx/d\theta} = \cancel{\dfrac{dr/d\theta\cos\theta + r\sin\theta}{dr/d\theta\sin\theta + r\cos\theta}}$

$$= \frac{dr/d\theta\,\sin\theta + r\cos\theta}{dr/d\theta\,\cos\theta - r\sin\theta} = \boxed{\frac{\dot{r}\sin\theta + r\cos\theta}{\dot{r}\cos\theta - r\sin\theta}}$$

Horizontal tangent lines occur when $\dot{y}=0$ (and $\dot{x}\neq 0$), and vertical tangents when $\dot{x}=0$ (but $\dot{y}\neq 0$).

tangents at the pole (where $r=0$) are given by $\frac{dy}{dx}=\tan\theta$ if $\dot{r}\neq 0$.

<u>Ex</u>. $r=\cos 2\theta$ (the 4-l. rose)

$r=0$ when $\theta=\frac{\pi}{4}, \frac{3\pi}{4}$, etc.

$$\left.\begin{aligned} &\text{at } \theta=\tfrac{\pi}{4}, \quad \frac{dy}{dx}=\tan\frac{\pi}{4}=1 \\ &\theta=\tfrac{3\pi}{4}, \quad \frac{dy}{dx}=\tan\frac{3\pi}{4}=-1 \end{aligned}\right\}$$ this matches the picture! ☺

11/12/12

<u>Ex</u>. Find the slope of the tan line $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$ for the cardiod $r=1+\sin\theta$.

$\dot{r}=\cos\theta$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos\theta\,\sin\theta+(1+\sin\theta)\cos\theta}{\cos^2\theta-(1+\sin\theta)\sin\theta}$$

$$=\frac{\cos\frac{\pi}{3}\sin\frac{\pi}{3}+\left(1+\sin\frac{\pi}{3}\right)\cos\frac{\pi}{3}}{\left(\cos\pi/3\right)^2-\left(1+\sin\frac{\pi}{3}\right)\sin\frac{\pi}{3}}$$

$$=\frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+\left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2-\left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}=\frac{\frac{\sqrt{3}}{4}+\frac{2}{4}+\frac{\sqrt{3}}{4}}{\frac{1}{4}-\frac{2\sqrt{3}}{4}-\frac{3}{4}}$$

$$=\frac{2\sqrt{3}+2}{-2-2\sqrt{3}}=\frac{\sqrt{3}+1}{-(\sqrt{3}+1)}$$

$$=\boxed{-1}\ !$$

Ex. 55! Show that $r = a\sin\theta + b\cos\theta$, $ab \neq 0$, represents a circle. Find its center and radius.

$$r^2 = a\, r\sin\theta + b\, r\cos\theta$$

$$x^2 + y^2 = ay + bx$$

$$x^2 - bx + \left(\frac{b}{2}\right)^2 + y^2 - ay + \left(\frac{a}{2}\right)^2 = 0 + \left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2$$

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{b^2 + a^2}{4}$$

so $C = \left(\frac{b}{2}, \frac{a}{2}\right)$ and $r = \frac{\sqrt{a^2+b^2}}{2}$. WORD!

Ex. 45. Show that the curve $r = \sin\theta\tan\theta$ (Cissoid of Diocles) has a v. asymptote at $x = 1$, and the curve lies entirely within the v. strip $0 \leq x < 1$.

$$r \cdot r = r\sin\theta \frac{r\sin\theta}{r\cos\theta}$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y - xy}$$

$$x^2 + y^2 = y \cdot \frac{y}{x}$$

$$x^2 + y^2 = \frac{y^2}{x} \longrightarrow y^2(1-x) = x^3$$

$$y = \pm\sqrt{\frac{x^3}{1-x}} \longleftarrow \text{V.A.}$$

$$\frac{d}{dx}\left[x^3 + xy^2 = y^2\right]$$

$$3x^2 + y^2 + xy\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[2y - xy\right] = 3x^2 + y^2$$

$$\frac{x^3}{1-x} \geq 0$$

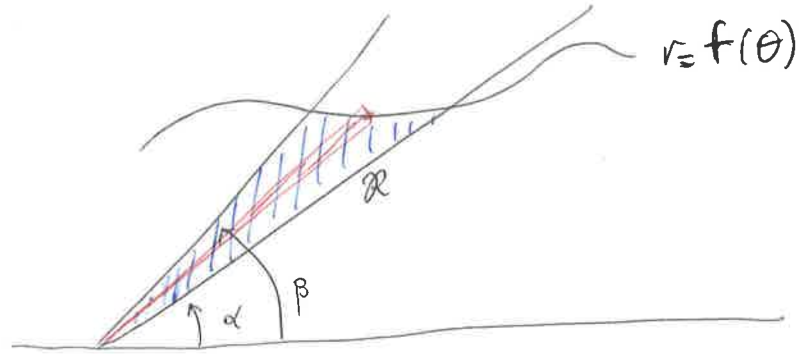
$$x^3 \geq 1 - x$$

only when $0 \leq x < 1$

Sign chart.

Areas and lengths of Polar curves:

$r = f(\theta) = r(\theta)$



recall

$A = \frac{1}{2} r^2 \theta$

Area of a sector of a circle.

so Area of $\mathcal{R}$ is approximately:

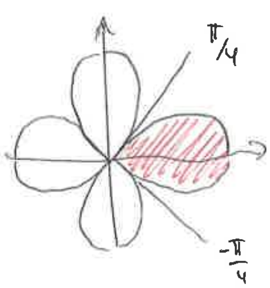
$$A \approx \sum_{n=1}^{m} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta \qquad \text{"Riemann Sum"}$$

and the actual area is given by:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta = \boxed{\int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 d\theta}$$

Ex. Find the area of on petal of the 4-leaved rose;

$r = \cos 2\theta$



$$A = \int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2}\cos^2 2\theta \, d\theta = 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

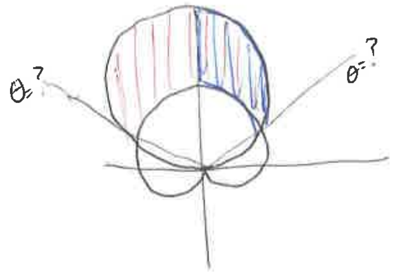
$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{4}\cdot\frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{8}}$$

Ex. Find the area inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1+\sin\theta$.



$3\sin\theta = 1+\sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

$A = \frac{1}{2}\int_\alpha^\beta r_o^2 - r_i^2\, d\theta$

$\cancel{A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta - 1 - \sin\theta)^2\, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta - 1)^2\, d\theta}$

$\cancel{= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4\sin^2\theta - 4\sin\theta + 1\, d\theta}$

$r_o = 3\sin\theta \Rightarrow r_o^2 = 9\sin^2\theta$

$r_i = 1+\sin\theta \Rightarrow r_i^2 = 1 + 2\sin\theta + \sin^2\theta$

$r_o^2 - r_i^2 = 8\sin^2\theta - 2\sin\theta - 1$

$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\sin\theta - 1\, d\theta$

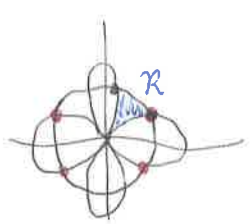
$= \int_{\pi/6}^{\pi/2} -4\cos 2\theta - 2\sin\theta + 3\, d\theta$

$= 3\theta + 2\cos\theta - 2\sin 2\theta \Big|_{\pi/6}^{\pi/2}$

$= \frac{3\pi}{2} - \frac{3\pi}{6} + 2\cos\pi/2 - 2\cos\pi/6 - 2\sin\pi + 2\sin\frac{\pi}{3}$

$= \pi + 0 - \cancel{\sqrt{3}} + 0 + \cancel{\sqrt{3}} = \boxed{\pi}$ phew.

Ex. Find all points on the intersection of $r = \cos 2\theta$ and $r = \frac{1}{2}$, then find A of R.



$\cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

but this only gives 4 of the 8 intersections.

There are 4 other intersection points: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

These come from setting $\cos 2\theta = -\frac{1}{2}$.

We want

$$A(R) = \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/3} \frac{1}{4} - \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{4}\theta\Big|_{\pi/6}^{\pi/3} - \frac{1}{4}\int_{\frac{\pi}{6}}^{\pi/3} 1 + \cos 4\theta \, d\theta$$

$$= \frac{1}{8}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{8}\left(\frac{\pi}{6}\right) - \frac{1}{4}\left(\frac{\pi}{6}\right) - \frac{1}{16}\left(\sin\frac{4\pi}{3} - \sin\frac{4\pi}{6}\right)$$

$$= -\frac{\pi}{48} - \frac{1}{16}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{3\sqrt{3}-\pi}{48}}$$

Arc Length:

Same trick as we used w/ derivates (slope of tan lines)

$$\left.\begin{aligned} x &= r(\theta)\cos\theta \\ y &= r(\theta)\sin\theta \end{aligned}\right\} \text{ both parametrized by } \theta$$

recall the parametrized formula for arc length

$$L_\gamma(a,b) = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\, d\theta$$

$$\dot{x} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \Rightarrow (\dot{x})^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$\dot{y} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \Rightarrow (\dot{y})^2 = \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$+$

$$\left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

So arc length becomes:

ehh, not a parameter for r.

$$L\gamma(a,b) = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + (\dot{r})^2}\, d\theta$$

Ex. Find the perimeter of the cardioid: $r = 1 + \sin\theta$

$$r^2 = 1 + 2\sin\theta + \sin^2\theta$$

$$\frac{dr}{d\theta} = \cos\theta \qquad \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + 2\sin\theta + \sin^2\theta + \cos^2\theta = 2 + 2\sin\theta$$

$$L(0, 2\pi) = \int_0^{2\pi} \sqrt{2(1+\sin\theta)}\, d\theta$$

$$= \int_0^{2\pi} \frac{2(1+\sin\theta)}{\sqrt{2+2\sin\theta}}\, d\theta \quad \text{ew!}$$

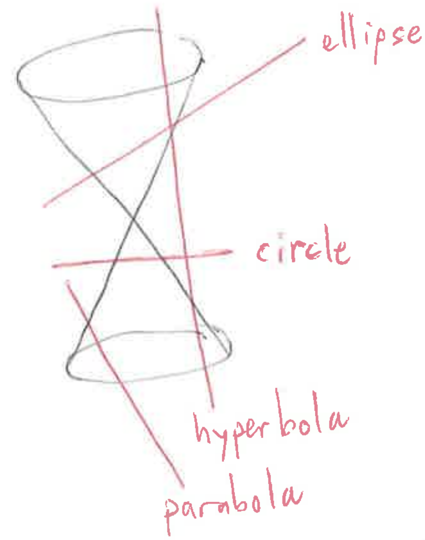
Wolfram|Alpha: $\boxed{L = 8}$.

Hand out College Algebra Conic Section worksheet.

§9.5: Conic Sections

Review handout:

4 types of conic sections



These all have Cartesian equations. (See Project.): if $(h,k)=(0,0)$

Circle: $x^2+y^2=r^2$ OR $\boxed{\dfrac{x^2}{r^2}+\dfrac{y^2}{r^2}=1}$

"Locus of points" definitions...

Parabola: $\boxed{y=ax^2}$

Ellipse: $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$, $a \geq b > 0 \Rightarrow$ horizontal

Hyperbola: $\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$

Exc. Write these out for $(h,k) \neq (0,0)$.

Our job today is to write these in Polar coordinates.

Thm. Let F be a fixed point (called the focus) and ℓ be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points in the plane such that

$$\frac{d(P,F)}{d(P,\ell)} = e$$

is a conic section. The conic is

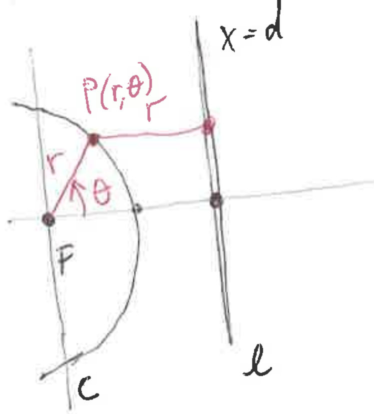
i.) an ellipse if $e<1$

ii) a parabola if $e=1$

iii) a hyperbola if $e>1$

Proof. If $e=1$, then this is just the defn of a parabola.

For the other two, we might as well put $F=(0,0)$ and make ℓ parallel to the y-axis, say $x=d$.



$P(r,\theta)_p = P(r\cos\theta, r\sin\theta)_c$

Put $d(P,F)=r$ so that $d(P,\ell)=d(P,Q)=d-r\cos\theta$

Then $$\frac{d(P,F)}{d(P,\ell)} = \frac{r}{d-r\cos\theta} = e$$

so $$r = e(d-r\cos\theta) \quad (*)$$

Square both sides of (*):

$$r^2 = e^2(d - r\cos\theta)^2 = e^2(d-x)^2 = e^2d^2 - 2e^2dx + e^2x^2$$

$$r^2 = e^2d^2 - 2e^2d\, r\cos\theta + r^2\cos^2\theta$$

$$\Rightarrow \quad x^2 + y^2 = e^2d^2 - 2e^2dx + e^2x^2$$

$$\underbrace{(1-e^2)x^2 + 2e^2dx}_{\text{complete the square!}} + y^2 = e^2d^2$$

$$x^2 + \frac{2e^2d}{1-e^2}x + \frac{e^4d^2}{(1-e^2)^2} + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2} + \frac{e^4d^2}{(1-e^2)^2}$$

$$b = \frac{2e^2d}{1-e^2}, \quad \frac{b}{2} = \frac{e^2d}{1-e^2}, \quad \left(\frac{b}{2}\right)^2 = \frac{e^4d^2}{(1-e^2)^2}$$

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{(1-\cancel{e^2})e^2d^2 + \cancel{e^4d^2}}{(1-e^2)^2}$$

$$\Rightarrow \quad \left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

multiply through by $\frac{(1-e^2)^2}{e^2d^2}$ (or divide by $\frac{e^2d^2}{(1-e^2)^2}$) to get

$$\frac{\left(x + \frac{e^2d}{1-e^2}\right)^2}{\frac{e^2d^2}{(1-e^2)^2}} + \frac{y^2}{\frac{e^2d^2}{(1-e^2)}} = 1$$

Suppose $e \gtrless 1$ here

if we put $a^2 = \frac{e^2d^2}{(1-e^2)^2}$, "$b^2$" $= \frac{e^2d^2}{1-e^2}$, $h = \frac{-e^2d}{1-e^2}$, then

this is

$$\frac{(x-h)^2}{a^2} \text{ "+" } \frac{y^2}{b^2} = 1 \quad \text{when } e<1$$

and

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{when } e>1.$$

In Cartesian coord's, eccentricity of an ellipse is given by $e = c/a$; where c is the distance of the focus from the center.

By right Δ trig, $c^2 = a^2 - b^2$

$$= \frac{e^2d^2}{(1-e^2)^2} - \frac{e^2d^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2} - \frac{(1-e^2)e^2d^2}{(1-e^2)^2}$$

$$= \frac{e^4d^2}{(1-e^2)^2}$$

So $c = \dfrac{e^2d}{1-e^2} = -h$ ✓

and $e = \dfrac{e^2d/1-e^2}{ed/1-e^2} = \dfrac{e^2d}{ed} = e$ ✓.

If $e<1$, get similar results for hyperbola.

For a hyperbola $c^2 = a^2 + b^2$.

We can use eq'n (*) to write polar equations for ellipses and hyperbolas:

$$(*) \quad r = e(d - r\cos\theta)$$

Solve for r: $r = ed - re\cos\theta$

$r + re\cos\theta = ed$

$$\boxed{r = \frac{ed}{1+e\cos\theta}} \quad \begin{cases} \text{parabola if } e=1 \\ \text{ellipse if } e<1 \\ \text{hyperbola if } e>1 \end{cases}$$

Also, we can replace $\cos\theta$ w/ $\sin\theta$ to get

$$\boxed{r = \frac{ed}{1+e\sin\theta}}$$ same e restrictions.

Ex. Find a polar equation for a parabola that has its focus at the origin and whose directrix is $y = -6$.

Sol'n: $e = 1$, $d = 6$, which of the eqns to use?

Use Wolfram to plot them all.

See that

$$r = \frac{6}{1 - \sin\theta}$$

directrix determines $\pm(\sin/\cos)\,\theta$

is the one.

Ex. $r = \dfrac{10}{3 - 2\cos\theta}$

Find the eccentricity, locate directrix, find other focus.

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta} = \frac{\overset{e}{\frac{2}{3}}\,(\overset{d}{})\,5}{1 - \frac{2}{3}\cos\theta}$$

$e = 2/3$, directrix: $x = -5$.

vertices when $\theta = 0, \pi$

$$r(0) = \frac{10}{3 - 2(1)} = \frac{10}{1} = 10$$

$$r(\pi) = \frac{10}{3 - 2(-1)} = \frac{10}{5} = 2$$

~~center at r=6~~

$(4,0)$ center.

other focus at $(8,0)$!

Ex. $r = \dfrac{12}{2+4\sin\theta}$ hyperbola

Now: Back to Ch. 10.