Chapter 8: Series

8.1 : Sequences

A sequence is a function $a(n) = a_n$ whose domain is

$$\mathbb{N} = \{ 1, 2, 3, 4, 5, \ldots \}$$

or

A sequence is an ordered list of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

we write either (a_n) or $(a_n)_{n=1}^{\infty}$

Ex. $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+1}$ $\quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \right\}$

$\left(\sqrt{n-3}\right)_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3}, n \geq 3$ $\quad \{ 0, 1, \sqrt{2}, \sqrt{3}, 2, \ldots, \sqrt{n-3}, \ldots \}$

this gives the same sequence as

$$a_n = \sqrt{n}, \ n \geq 0$$

Ex. Find a formula for the general term

a.) $\left\{ -\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots \right\}$ $\quad a_n = (-1)^n \frac{n}{(n+1)^2}$

b.) $\{ 2, 7, 12, 17, \ldots \}$ $\quad a_n = 2 + 5(n-1) = -3 + 5n$

c.) $\{ 5, 1, 5, 1, 5, 1, \ldots \}$ $\quad a_n = \begin{cases} 5 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$

Some sequences don't have nice formulas.

Ex. The Fibonacci Sequence

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

$\left.\begin{array}{l} a_1 = 1 \\ a_2 = 1 \\ a_{n+2} = a_{n+1} + a_n \end{array}\right\}$ recursive definition: the next one depends on the previous.

Ex. $a_n =$ n^{th} decimal place of the natural number

$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$

~~[illegible]~~ ~~[illegible]~~
Never "repeats" because e is irrational.

We want to do calculus on/with sequences!

Def'n. A sequence (a_n) has the limit L and we write

$$\lim_{n\to\infty} a_n = \lim a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if and only if for every $\varepsilon > 0$ there exists an integer N such that if $n > N$, then $|a_n - L| < \varepsilon$.

i.e., we can make the terms a_n as close to L as we want by taking a large enough n.

If (a_n) has a limit we say the sequence converges.

Otherwise, it diverges.

Ex. Prove that $a_n = \frac{n}{2n+1}$ has limit $\lim a_n = \frac{1}{2}$.

Fix $\varepsilon > 0$, if $n > N = \_\_\_\_$ then

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n - 2n - 1}{4n+2}\right| = \left|\frac{1}{2(2n+1)}\right| < \varepsilon$$

$2n+1 > \frac{1}{2\varepsilon}$

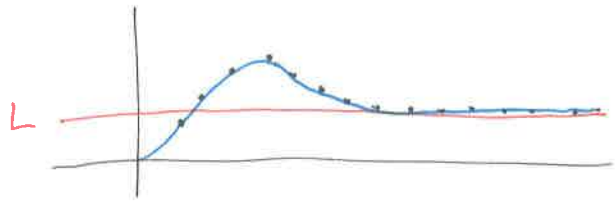
$2n > \frac{1}{2\varepsilon} + 1$

$n > \frac{1}{4\varepsilon} + \frac{1}{2}$

$= \frac{2\varepsilon + 1}{4\varepsilon}$

$>$

Thm. If f is a real function ~~and~~ s.t. $f(n) = a_n$, and $\lim_{x\to\infty} f(x) = L$, then $\lim a_n = L$.



In particular, $\lim_{n\to\infty} \frac{1}{n^r} = 0$ if $r > 0$.

Defn. $\lim a_n = \infty$ means that for every positive number M, there is an integer N such that if

$$n > N, \text{ then } a_n > M$$

Such a limit diverges "to infinity".

Limit rules: If (a_n) and (b_n) are convergent sequences and $c \in \mathbb{R}$, then

1. $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$
2. $\lim (c\, a_n) = c \lim a_n$
3. $\lim (a_n \cdot b_n) = \lim a_n \cdot \lim b_n$
4. $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if $\lim b_n \neq 0$.
5. $\lim a_n^p = [\lim a_n]^p$ if $p > 0$ and $a_n > 0$

Squeeze Thm. If $a_n \le b_n \le c_n$ for $n \ge N$, ~~then~~ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Ex. Calculate $\lim \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$

Ex. Convergent or divergent?

$(a_n) = \{-1, 1, -1, 1, -1, 1, -1, 1, \dots\}$

D.

~~Ex~~ Theorem. If $\lim |a_n| = 0$, then $\lim a_n = 0$.

Ex. Evaluate $\lim \frac{(-1)^n}{n}$

$\lim \left| \frac{(-1)^n}{n} \right| = \lim \frac{1}{n} = 0$, so $\lim \frac{(-1)^n}{n} = 0$ also.

Ex. $a_n = \frac{n!}{n^n}$ Converge or diverge?

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$n^n = n \cdot n \cdot n \cdots n$

so $a_n = \frac{1}{n} \left(\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n} \right)$ looks like these are decreasing to 0?

this ≤ 1

so $0 < a_n \leq \frac{1}{n}$

Now, use squeeze thm! get $\lim a_n = 0$. ☺

Ex. $\lim r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \end{cases}$ diverges if $r > 1$.

Defn. A sequence is called increasing if $a_{n+1} > a_n$ for all n. decreasing if $a_{n+1} < a_n$ for all n. Monotonic if it's either inc or dec.

Ex. Show that $a_n = \frac{3}{n+5}$ is decreasing

$$a_{n+1} = \frac{3}{n+1+5} = \frac{3}{n+6} < \frac{3}{n+5} = a_n \quad \text{for all } n.$$

Ex. $a_n = \frac{n}{n^2+1}$ $\quad a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+1+1}$ dec.

Cross multiply to get

$$n^3+2n^2+2n \overset{?}{>} (n+1)(n^2+1) = n^3+n^2+n+1$$

$$n^2+n > 1 \quad \checkmark \quad \text{since } n \geq 1.$$

Defn. A sequence is bounded above if $\exists M$ s.t.

$$a_n \leq M \quad \text{for all } n.$$

bounded below if $\exists m$ s.t.

$$m \leq a_n \quad \text{for all } n.$$

If $m \leq a_n \leq M$, then a_n is bounded.

Theorem. Monotone Sequence Thm. Every bounded, monotonic sequence is convergent.

Proof. In the book. Justify w/ a picture.

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limits:

Ex. $a_n = \left(1+\frac{2}{n}\right)^n$

// 10/10/12

8.2. Series

Let $(a_n) = (a_n)_{n=1}^{\infty}$ be a sequence. If we add up all of the terms we get

$$a_1 + a_2 + a_3 + \dots + a_n + \dots =: \sum_{n=1}^{\infty} a_n = \sum a_n$$

Does it make sense to add up infinitely many things?

This is called an <u>infinite series</u>.

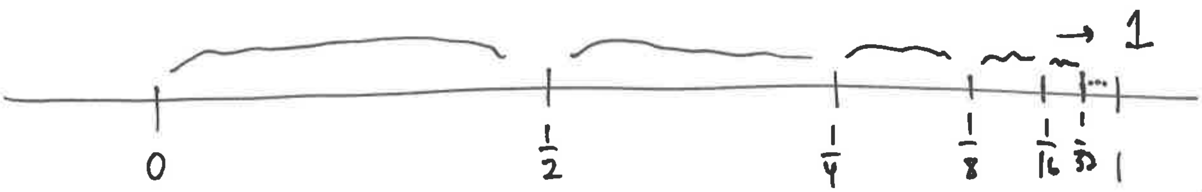
<u>Ex</u>. $1 + 2 + 3 + 4 + 5 + \dots + n + \dots$

We get ∞ (not a number), so we say that the series $\sum_{n=1}^{\infty} n$ diverges.

<u>Ex</u>. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$

We add smaller and smaller terms. This is a <u>geometric series</u> (from HS algebra). We know that the sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}(2) = 1.$$



<u>Idea</u>:

Let $s_1 = a_1$

$s_2 = a_1 + a_2 = s_1 + a_2$

$s_3 = a_1 + a_2 + a_3 = s_2 + a_3$

$\vdots$

$s_n = a_1 + \dots + a_n = s_{n-1} + a_n$

These get closer and closer to 1.

We write $s_n = \sum_{i=1}^{n} a_i$ and call this the <u>n^{th} partial sum</u> of the series $\sum a_n$.

Notice that $s_\infty = \sum_{i=1}^{\infty} a_i = S$ is the sum of the whole series, if it exists.

We write:

$$\boxed{S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i}$$

If this limit exists, S is called the <u>sum</u> of the series. If not, we say the series diverges.

Prop. The geometric series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

is convergent if $|r|<1$, and its sum is

$$S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad , \ |r|<1$$

If $|r| \geq 1$, then the geometric series diverges.

<u>Proof</u>. Let $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$

then $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

and shift.

$S_n - rS_n = a - ar^n$

or $\boxed{S_n = \dfrac{a(1-r^n)}{1-r}}$

So, the n^{th} partial sum is given by

$$s_n = \frac{a(1-r^n)}{1-r}$$

then $$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a\left(1-\lim_{n\to\infty} r^n\right)}{1-r}$$

$$= \frac{a}{1-r} \quad \text{if } |r|<1 \quad \text{and}$$

$$\text{diverges if } |r| \geq 1$$

If $r=1$ or $r=-1$, then it's easy to see that

$$\sum_{n=1}^{\infty} a r^{n-1} \text{ diverges.} \qquad \square$$

<u>Ex</u>. Find the sum

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + - \cdots$$

$$= \sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^{n-1} = \frac{5}{1+\frac{2}{3}} = \frac{5}{5/3} = \boxed{3}$$

<u>Ex</u>. $$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} \qquad \text{conv. or div.?}$$

$$= \sum_{n=1}^{\infty} 4^n \cdot 3 \cdot 3^{-n}$$

$$= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{4}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1} \qquad r = \frac{4}{3} > 1, \text{ so } \underline{\text{div.}}$$

Ex. Write the number $2.3\overline{17} = 2.3171717717\ldots$ as a decimal.

$$2.3\overline{17} = 2.3 + .017 + .00017 + .0000017 + \cdots$$

$$= 2.3 + 17(10^{-3}) + 17(10^{-5}) + 17(10^{-7}) + \cdots$$

$$= 2.3 + \frac{17}{10^3} + \frac{17}{10^3}\cdot\frac{1}{10^2} + \frac{17}{10^3}\cdot\frac{1}{10^4} + \cdots$$

$$= 2.3 + \sum_{n=1}^{\infty} \frac{17}{10^3}\left(\frac{1}{10^2}\right)^{n-1}$$

$$= \frac{230}{100} + \frac{17}{1000}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= \frac{230}{100} + \frac{17\cdot 100}{99\cdot 1000} = \frac{23}{10} + \frac{17}{990} = \frac{99\cdot 23 + 17}{990}$$

$$= \frac{2294}{990} = \boxed{\frac{1147}{495}}$$

Ex. Find the sum of the series $\sum_{n=0}^{\infty} x^n$, $|x|<1$.

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

This is a function of x w/ domain $(-1,1)$!

Ex. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{n(n+1)}$$

Ew.

Insted: PFD $\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1} = \frac{1}{i} - \frac{1}{i+1}$

Add this insted:

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

"telescoping" sum.

And $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$

__Ex.__ Show that the __harmonic series__ $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(p.424)

__RE__

get, in general, $S_{2^n} > 1 + \frac{n}{2}$

$\lim S_n = \lim S_{2^n}$ diverges!

__RE__: __Theorem.__ $\sum_{n=1}^{\infty} a_n$ is convergent implies $\lim_{n\to\infty} a_n = 0$.

__Proof.__ $S_n = a_1 + \cdots + a_n$

$S_{n+1} = a_1 + \cdots + a_{n+1}$

$\Rightarrow a_{n+1} = S_{n+1} - S_n$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - S_n = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = S - S = 0. \quad \square$$

__Test for divergence__: If $\lim_{n\to\infty} a_n =$ dne or $\neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

__Ex.__ Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$\lim a_n = \frac{1}{5} \neq 0.$

10/12/12

Thm. Spse $\sum a_n$ and $\sum b_n$ are conv. series. Then so are the series:

i) $\sum c a_n = c \sum a_n \qquad c \in \mathbb{R}$.

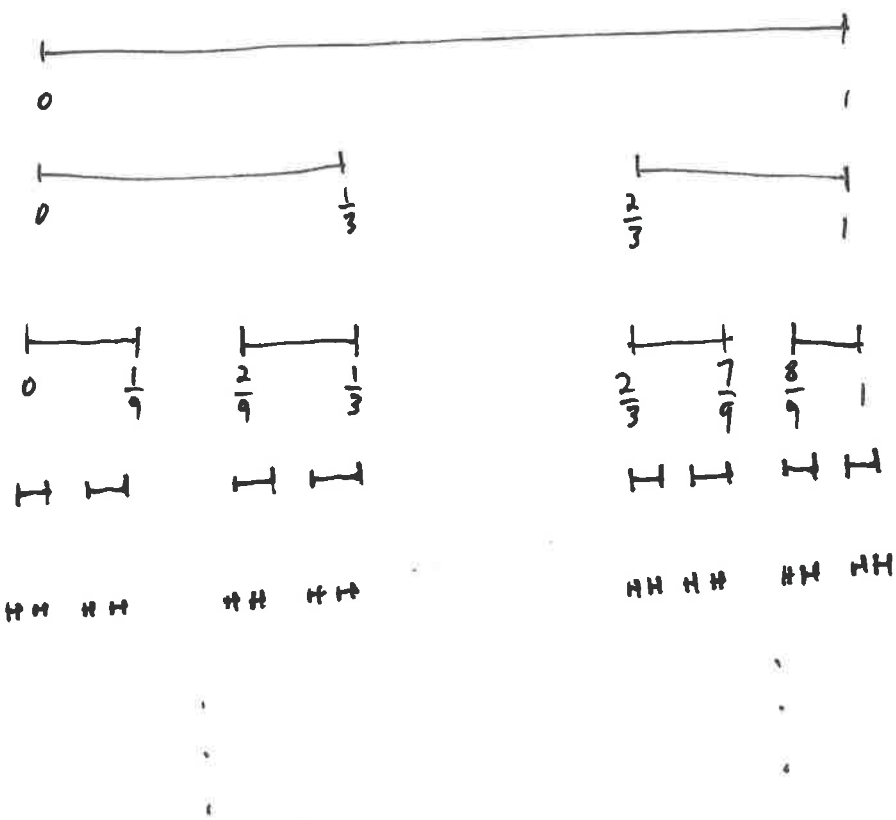
ii) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

Ex. Find the sum of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$$= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$= 3(1) + 1 = \boxed{4}$$

The Cantor set:



Step 1.

Step 2.

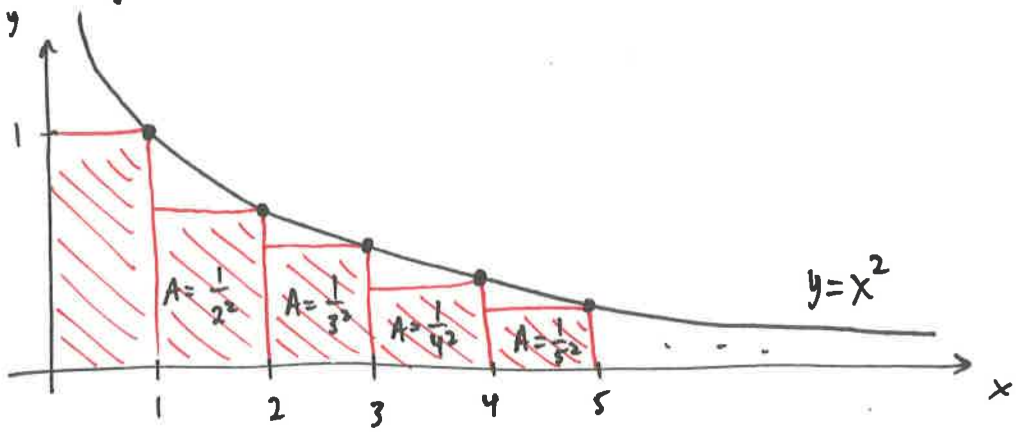
step 3.

Step 4.

Step 5.

8.3. Integral and Comparison Tests

Ex. We want to know whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.



The area of the first box is 1, which is obviously finite. If we ignore it, the rest of the boxes lie below the curve, so their areas add up to less than the integral of the function:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

So, if $\int_1^{\infty} \frac{1}{x^2}dx < \infty$, then so is $\sum_{n=1}^{\infty} \frac{1}{n^2}$!

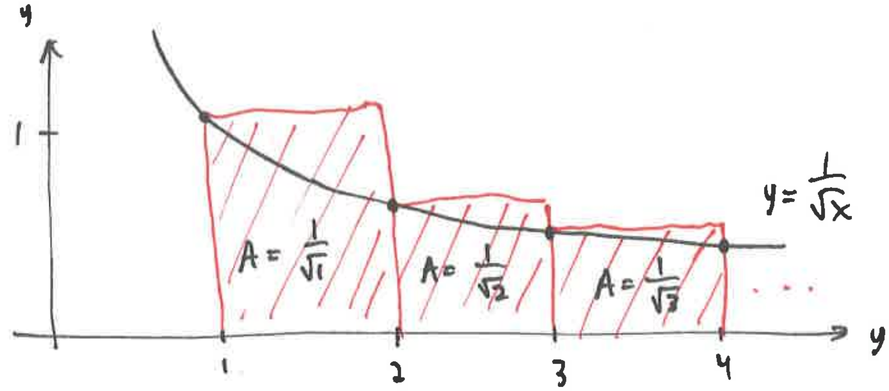
We know $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

We still don't know its exact sum, but this is a good start.

$\frac{\pi^2}{6}$!

Ex. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Use the same idea (kind of).



So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

We know that $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges, so the series must diverge as well.

This leads us to the Integral Test:

Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ such that $a_n = f(n)$ for $n \in \mathbb{N}$. Then $\Sigma(a_n)$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent, i.e.,

a. If $\int_1^{\infty} f(x)\,dx < \infty$, so is Σa_n.

b. If $\int_1^{\infty} f(x)\,dx$ is divergent, so is Σa_n.

* Note: We don't actually need to start at 1. Since partial sums are always finite, we only need to check a "tail" of the series. We can choose any lower limit that is "convenient."

Ex. Conv. or div.? : $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ ← decreasing? check via $\frac{d}{dx}$.

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} (\ln t)^2 = \infty,$$

So the series diverges.

Remember the p-test for integrals?

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$, diverges otherwise.

This gives us a p-test for series:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges otherwise (≤ 1)

Testing by Comparison.

Consider $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. Convergent or divergent?

$$0 < \frac{1}{2^n+1} < \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}$$

We know $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, so $\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

This series is convergent.

We have the Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series w/ positive terms, and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

a) If $\sum b_n$ is convergent, then so is $\sum a_n$

b) If $\sum a_n$ is divergent, then so is $\sum b_n$.

Ex. C or D: $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} =: \sum a_n$

$$0 < \frac{5}{2n^2+4n+3} < \frac{5}{2} \cdot \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}$$

and $\displaystyle\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum a_n$ also converges!

Ex. back to $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. We just showed that this div. w/ integral test. Also can use comp. test:

$$\frac{1}{n} < \frac{\ln n}{n} \quad \forall\, n \geq 3,$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum \frac{\ln n}{n}$ must also.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

try to compare to $\sum \frac{1}{2^n}$, which is convergent, but

$$\frac{1}{2^n-1} > \frac{1}{2^n} \quad \forall\, n \in \mathbb{N},$$

so the comparison test tells us nothing useful.

Instead,

The Limit Comparison Test:

Suppose $\sum a_n$, $\sum b_n$ are series w/ positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \quad (\text{in particular, } \neq 0)$$

then either both series converge, or both diverge.

Proof. let $m, M > 0$ such that $m < c < M$. since $\lim \frac{a_n}{b_n} = c$, then $\exists\, N \in \mathbb{N}$ s.t.

$$m < \frac{a_n}{b_n} < M \qquad \forall\, n > N$$

$$\Rightarrow \quad m b_n < a_n < M b_n$$

If $\sum b_n$ converges, so does $M \sum b_n$, thus $\sum a_n$ converges by the comparison test. If $\sum b_n$ diverges, then so does $m \sum b_n$, and $\sum a_n$ diverges by the comparison test. ∎

Ex. Use LCT to show $\sum \frac{1}{2^n - 1}$ converges.

Compare to $\sum \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{x\to\infty} \frac{2^x}{2^x - 1} \overset{L'H}{=} \lim_{x\to\infty} \frac{\ln(2)\, 2^x}{\ln(2)\, 2^x} = 1 > 0$$

$\therefore \sum \frac{1}{2^n - 1}$ converges.

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17 oct

RE 25. C or D: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

compare to $\sum \frac{1}{n}$ via LCT

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \overbrace{\lim_{u\to 0} \frac{\sin u}{u}}^{\text{F.T.I.S.}} = 1.$$

$\sum \frac{1}{n}$ diverges, so $\sum \sin\left(\frac{1}{n}\right)$ also diverges!

8.4. Other Convergence Tests

The **Alternating Series Test**:

If the alt. series $\sum (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + - \cdots$ $\quad (b_n > 0)$

satisfies $\quad$ i.) $b_{n+1} \leq b_n$ $\quad$ for all n

and ii.) $\lim b_n = 0$

then the series is convergent.

Figure 2. on pg. 439 illustrates why this works for the alternating harmonic series.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\quad$ so $b_n = \frac{1}{n}$

$b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$ $\quad$ for all $n \geq 1$.

also $\lim \frac{1}{n} = 0$, therefore the series converges by AST.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \, 3n}{4n-1}$, $\quad b_n = \frac{3n}{4n-1}$, $\quad \lim b_n = \frac{3}{4} \neq 0$, so diverge. by TFD.

Ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$, $\quad b_n = \frac{n^2}{n^3+1}$, $\quad \lim b_n = 0$, but is $\frac{n^2}{n^3+1}$

decreasing? $\quad (b_n)' = \frac{(n^3+1)(2n) - (n^2)(3n^2)}{(n^3+1)^2} = \frac{2n^4 + 2n - 3n^4}{(n^3+1)^2}$

$= \frac{2n - n^4}{(n^3+1)^2}$ $\quad$ for $n \geq 2$ this is decreasing. (neg.)

so $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is conv. by AST.

A.S. Estimation Thm.

If $\sum(-1)^n b_n$ is a conv. alt. series w/ $0 \le b_{n+1} \le b_n$ and $\lim b_n = 0$, then

$$|R_n| = |s - s_n| \le b_{n+1}$$

RE: Proof. s lies between any two partial sums, so

$$|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}. \quad \square$$

Ex. Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to 3 decimal places.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + - \cdots$$

$$= \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + - \cdots$$

$$\frac{1}{5040} < \frac{1}{5000} = 0.0002$$

So, $|s - s_6| \le b_7 < 0.0002$

Add up the first 7: $s \approx 0.368$

Defn. A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent.

of course $-\sum|a_n| \le \sum a_n \le \sum |a_n|$, so $\sum a_n$ is also convergent in this case.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely conv. by p-test.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conv by AST, but is not abs. conv. (p-test).

Such a series is called conditionally convergent.

Ex. C or D: $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ $\quad a_n = \frac{\cos n}{n^2}$, $|a_n| = \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-test, so $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ conv. by CT.

$\therefore \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

The Ratio Test :

1. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then $\sum a_n$ is divergent.
   (or $= \infty$)
3. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Rat. test is inconclusive.

Ex. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ Test for abs. conv.

$$a_n = \frac{(-1)^n \, n^3}{3^n}$$

$$a_{n+1} = \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cancel{(-1)^{n+1}} (n+1)^3}{3^{\cancel{n+1}}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} \, n^3} \right|$$

$$= \left| \frac{n^3 + 3n^2 + 3n + 1}{3n^3} \right|$$

$\lim \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} < 1$, so abs. conv!

Ex. C or D: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n^n}{n!} \qquad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} \qquad = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \left(1 + \frac{1}{n}\right)^n = e > 1$$

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So the series diverges by the rat. test!

The Root Test:

1. If $\lim \sqrt[n]{a_n} = L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\lim \sqrt[n]{a_n} = L > 1$ (or $= \infty$), then $\sum a_n$ is divergent.
3. If $\lim \sqrt[n]{a_n} = 1$, then the root test is inconclusive.

Ex. Test $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ for conv.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$$

$$\lim \sqrt[n]{a_n} = \lim \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

So convergent!

✓ Some REs:

18. For what p is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ convergent? [All $p>0$.]

42.* Srinivasa Ramanujan: ($\sim$1910)

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103+26390n)}{(n!)^4 \; 396^{4n}} = \frac{1}{\pi} \quad !$$

1985: William Gosper used this to calculate the first 17 million digits of π!

8.5. Power Series

A power series is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and c_i are constants. (coefficients)

For any fixed x, this is a series of numbers that may or may not converge.

~~If the series converges, then~~ the power series is a function that we can think of as an "infinite polynomial."

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Ex. $\sum_{n=0}^{\infty} x^n = f(x) = 1 + x + x^2 + x^3 + \cdots$

This series converges when $-1 < x < 1$ (geometric series).

so f is a function w/ domain $(-1,1)$

For any $a \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} (x-a)^n = 1 + (x-a) + (x-a)^2 + (x-a)^3 + \cdots$$

converges for $a-1 < x < a+1$.

This is called a power series centered at a.

(Really put in the c_n's)

Ex. For what values does $\sum_{n=0}^{\infty} n!\,x^n$ converge?

Sol'n. Use the ratio test.

$$a_{n+1} = (n+1)!\,x^{n+1} = (n+1)\,x\;n!\,x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\,x\,\cancel{n!}\,\cancel{x^n}}{\cancel{n!}\,\cancel{x^n}}\right| = |x|\,(n+1)$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim |x|\,(n+1) = \infty$$

So this series diverges for all $x \neq 0$, but converges for $x = 0$.

Ex. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ For what values of x does this converge?

$$a_{n+1} = \frac{(x-3)^n\,(x-3)}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(x-3)^n}\,(x-3)}{n+1} \cdot \frac{n}{\cancel{(x-3)^n}}\right| = \frac{n}{n+1}\,|(x-3)|$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n}{n+1}\,|x-3| = |x-3|$$

By the Ratio Test, this series will converge if the limit is < 1, or $|x-3| < 1$

$$\Rightarrow \quad -1 < x-3 < 1$$

$$2 < x < 4$$

we need to plug in the end points individually:

at $x=2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST.

$x=4$: $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum \frac{1}{n}$ diverges by p-test.

So Radius of Conv. is $\boxed{2 \leq x < 4}$

Ex. The 0^{th} Bessel function:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Find its domain.

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{\cancel{(-1)^{n+1}} x^{2n+2}}{2^{2n+2} ((n+1)!)^2} \cdot \frac{\cancel{2^{2n}} \cancel{(n!)^2}}{\cancel{(-1)^n} \cancel{x^{2n}}} \right|$$

$$= \lim \left| \frac{x^2}{4 (n+1)^2} \right| = 0$$

so $\sum$ converges for all x, and $D(J_0) = \mathbb{R}$!

Graph some Bessel partial Sums?

Ex. Find the radius of convergence R and the interval of convergence of:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$\lim \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n} \cancel{x^n}} \right| = \lim \left| \frac{-3\sqrt{n+1}}{\sqrt{n+2}} \right| |x| = 3|x|$$

need: $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$, so $R = \frac{1}{3}$

$$-\frac{1}{3} < x < \frac{1}{3}$$

By p-test and AST: $\boxed{-\frac{1}{3} < x \leq \frac{1}{3}}$

Ex. Find R and I of C:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

$$\lim \left| \frac{(n+1)\cancel{(x+2)^{n+1}}}{\cancel{3^{n+2}}} \cdot \frac{\cancel{3^{n+1}}}{n\,\cancel{(x+2)^n}} \right| = \frac{|x+2|}{3} \lim \left| \frac{n+1}{n} \right| = \frac{|x+2|}{3}$$

$$|x+2| < 3 \implies -5 < x < 1$$

$\underline{R = 3}$

Plug in -5: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n n$ diverges.

Plug in 1: $\displaystyle\sum_{n=0}^{\infty} \frac{n\,3^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} n$ diverges

so, I of C: $\boxed{-5 < x < 1}$

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8.6. Representing functions as power series

First, recall the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

That is, the function $f(x) = \frac{1}{1-x}$, $-1 < x < 1$, is equivalent to the power series above!

* Notice that we <u>must</u> specify the domain. These do not agree everywhere.

Ex. Similarly, represent $f(x) = \frac{1}{1+x^2}$ as the sum of a power series and find R, and IC (the domain).

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

this is a geometric series, so $R=1$.

$$-1 < x < 1$$

diverges at each end point.

Ex. $f(x) = \frac{1}{x+2}$ Same Question.

$$\frac{1}{2+x} = \frac{1}{2} \frac{1}{1+\frac{x}{2}} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

converges when $\frac{|x|}{2} < 1$, or $|x| < 2$ so $R = 2$.

Again, its geometric, so IC: $(-2,2)$.

Ex. $f(x) = \frac{x^3}{x+2}$

$$x^3 \cdot \frac{1}{x+2} = x^3 \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$$

This is still geometric, so $R=2$ and $IC = (-2,2)$.

Ex. 12. $f(x) = \dfrac{7x+1}{3x^2+2x-1}$

PFD: $3x^2+2x-1 = (3x-1)(x+1)$

$$\frac{7x+1}{3x^2+2x-1} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{5}{2}\frac{1}{3x-1} + \frac{3}{2}\frac{1}{x+1}$$

$$7x+1 = A(x+1) + B(3x-1)$$

$$-6 = B(-4) \Rightarrow B = \frac{3}{2}$$

$$\frac{7}{3}+1 = A\left(\frac{1}{3}+1\right) \Rightarrow \frac{10}{3} = \frac{4A}{3} \Rightarrow A = \frac{5}{2}$$

$$f(x) = \frac{-5}{2}\frac{1}{1-3x} + \frac{3}{2}\frac{1}{1+x}$$

$$= \boxed{\frac{-5}{2}\sum_{n=0}^{\infty}(3x)^n + \frac{3}{2}\sum_{n=0}^{\infty}(-x)^n}$$

$IC = \left(-\frac{1}{3}, \frac{1}{3}\right)$ b/c of the left half.

For 11. $f(x) = \dfrac{3}{x^2+x-2}$ also try completing the square.

$$x^2+x-2 = \left(x+\frac{1}{2}\right)^2 - \frac{9}{4}$$

so $f(x) = \dfrac{3}{(x+\frac{1}{2})^2 - \frac{9}{4}}$

$$= -\frac{4}{9}\cdot\frac{3}{1-\frac{4}{9}(x+\frac{1}{2})^2}$$

$$= -\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{4}{9}\left(x+\frac{1}{2}\right)^2\right]^n$$

compare w/ your solution using PFD. (i.e. check R and IC).

Differentiation and Integration of power series

If $\sum c_n (x-a)^n$ has radius of convergence $R>0$, then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is differentiable (hence also continuous) on the interval $(a-R, a+R)$, and

1.) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$$

2.) $$\int f(x)\,dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Equivalently,

1.) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$ and

2.) $$\int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx$$

That is, $\sum$ commutes w/ differentiation and integration.

Moreover, the radius of convergence is not changed (it's still R), but the end points may or may not converge any more.

Ex. The Bessel Function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}}$$

must remember to do this!

If you want to shift the index to start at 0 it will become:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cancel{2}(n+1) x^{2n+1}}{2^{2n+\cancel{2}1}((n+1)!)^2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1) x^{2n+1}}{2^{2n+1}((n+1)!)^2}}$$

Ex. Write $\frac{1}{(1-x)^2}$ as a power series.

Notice: $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[x^n] = \sum_{n=1}^{\infty} n x^{n-1}$

or, $\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n}$

Ex. Write a power series representation for $f(x) = \ln(1-x)$.

$\int \frac{1}{1-x}\,dx = -\ln(1-x)$, so $\ln(1-x) = -\int \frac{1}{1-x}\,dx$

again $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $-\int \frac{1}{1-x}\,dx = -\sum_{n=0}^{\infty} \int x^n\,dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$

To determine the value of C, we put in $x=0$, and get

$0 + C = 0$, so $C = 0$

then $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Plug in $x=\frac{1}{2}$ to get

$$\ln\left(\tfrac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

but $\ln\left(\frac{1}{2}\right) = -\ln(2)$, so

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n\,2^n}.$$

Ex. Find a power series representation for $f(x) = \tan^{-1}(x)$

$$f(x) = \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Put in $x=0$ to get $C=0$, so

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Ex. 35. Show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a solution to the DE:

$$f'(x) = f(x)$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Ex. 37. Add this as an RE

let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ Find IC for f, f', f''.

For f: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \cdot \frac{n^2}{n^2+2n+1} \to |x| < 1$ so $R = 1$

$IC = [-1, 1]$

$f'(x) = \sum_{n=2}^{\infty} \frac{n\, x^{n-1}}{n^2} = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{x^n}{n+1}$ $R=1$ still

$IC = [-1, 1)$

$f''(x) = \sum_{n=2}^{\infty} \frac{n\, x^{n-1}}{n-1} = \sum_{n=1}^{\infty} \frac{(n-1)\, x^n}{n-2}$ $R=1$ still

$IC = (-1, 1)$

Ex. Integrate: $\int \frac{1}{1+x^7}\, dx$

$$\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

So $\int \frac{1}{1+x^7}\, dx = \int \sum_{n=0}^{\infty} (-\cancel{x})^n x^{7n}\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{7n}\, dx$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} + C$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + - \cdots$$

Converges for $|x| < 1$.

Ex. 35!. $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Find R and I.C.

Show that $f'(x) = f(x)$.

Deduce that $f(x) = e^x$

Put $x=1$, and write out a series that adds up to e.

Ex. (21.) Find a P.S. rep. for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Ex. (20.) $f(x) = \frac{1}{x^2+25}$.

Find P.S. rep. and graph a few S_n.

Ex. (32.) Show that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ solves

$$f''(x) + f(x) = 0.$$

Ex. (34.)

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!\, 2^{2n+1}}$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Show that $J_0'(x) = -J_1(x)$.

Ex. Use a Power series to approximate the definite integral to six decimal places

$$\int_0^{0.3} \frac{x^2}{1+x^4}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{n+3} \Big|_0^{0.3}$$

$$\frac{x^2}{1+x^4} = \sum_{n=0}^{\infty} x^2 (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$

8.7. Taylor and Maclaurin Series

As we discussed on Friday: we can estimate a function by a polynomial if we insist that the function's derivatives agree w/ the polynomial's derivatives.

This leads to the idea of a Taylor series.

Thm. If f has a power series expansion at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (*)$$

Recall: In general a power series is given by:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

So this theorem gives a formula for the coefficients c_n.

Equation $(*)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots$$

is called the Taylor Series expansion of f at a.

If $a=0$, then $(*)$ becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

is called the Maclaurin Series of f.

Ex. Find the MacLaurin Series for $f(x)=e^x$.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$\left.\begin{aligned} f(0)&=e^0=1\\ f'(0)&=e^0=1\\ f''(0)&=e^0=1\end{aligned}\right\}\text{ So } c_n=\frac{1}{n!}\Rightarrow \boxed{f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}x^n}$$

We found on Friday that the radius of convergence is ∞.

Question: how do we know that this series actually converges to $f(x)$?

Let $T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$ be the n^{th} partial sum.

This is called the n^{th} Taylor polynomial of f.

Then $R_n(x)=f(x)-T_n(x)$ is the Remainder of the Taylor Series.

So, $f(x)=T_n(x)+R_n(x)$ for any n.

Therefore, a Taylor series for f converges to f iff

$$\lim_{n\to\infty}R_n(x)=0.$$

We get:

Taylor's Formula: If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I, there is a number z strictly between x and a such that

$$\boxed{R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}}$$

Lagrange's form

Ex. Prove that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$f^{(n+1)}(x) = e^x$, so

$$R_n(x) = \frac{e^z}{(n+1)!} x^{n+1}$$

where $z \in (0, x)$.

(*) Note that z depends on x and on n.

If $x > 0$, then $0 < z < x$, so $e^z < e^x$ and

$$0 < R_n(x) = \frac{e^z}{(n+1)!} x^{n+1} < e^x \frac{x^{n+1}}{(n+1)!} \longrightarrow 0.$$

Then $R_n(x) \to 0$ by the squeeze theorem.

If $x < 0$, then $x < z < 0$, so $e^z < e^0 = 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0.$$

Therefore e^x is equal to its Taylor Series expansion. □

Ex. Find the Taylor Series expansion of $f(x) = e^x$ at $x = 2$.

$f^{(n)}(x) = e^x$ so $f^{(n)}(2) = e^2$

so $\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n}$ for all x.

This is a better approx. "near" $x = 2$.

Ex. Find MacLaurin Series for $\sin x$.

$n=0 \quad f(x) = \sin x \quad f(0) = 0$

$n=1 \quad f'(x) = \cos x \quad f'(0) = 1$

$n=2 \quad f''(x) = -\sin x \quad f''(0) = 0$

$n=3 \quad f^{(3)}(x) = -\cos x \quad f^{(3)}(0) = -1$

$n=4 \quad f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$

We get a MacLaurin Series:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

To find R, use the Remainder Thm:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, either way

$|f^{(n+1)}(x)| \le 1$, so

$$|R_n(x)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \forall x.$$

So $R = \infty$, and $\sin x$ equals its Taylor Series expansion.

Ex. Find the MacLaurin Series for $\cos x$.

$$\cos x = \frac{d}{dx}\left[\sin x\right] = \frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots\right]$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)\,(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - + \cdots$$

Ex. Find MacLaurin Series for $x\cos x$.

Ex. Write e^{-x^2} as a power series.

1. Write MacLaurin
2. Integrate.

Ex. $f(x) = x^3$ around $a = -1$

Ex. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

Ex. Binomial Series.

Find Maclaurin series for $f(x) = (1+x)^k$

$$f(x) = (1+x)^k \qquad f(0) = 1$$
$$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$$
$$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0) = k(k-1)$$
$$f'''(x) = k(k-1)(k-2)(1+x)^{k-2} \qquad f'''(0) = k(k-1)(k-2)$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n+1} \qquad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

So M.S. is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Fact: ratio test: $R = 1$.

$$= \sum_{n=0}^{\infty} \underbrace{\frac{k!}{(k-n)!\,n!}}_{\binom{k}{n}} x^n$$

"k choose n" — binomial coefficients.

So the binomial series is given by

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for any } k \in \mathbb{R}, \text{ and } |x| < 1.$$

Ex. Find M.S. for $f(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-1/2} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}$

$$= \frac{1}{2}\left(1 - \left(-\frac{x}{4}\right)\right)^{-1/2}$$

So $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n}$ $\boxed{R = 4}$

Ex. Evaluate $\int e^{-x^2}dx$ as a power series

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

so $$\int e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C}$$

Ex. 54. Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ using taylor series.

First: write the first 3 terms of $\tan x$ expansion.

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots}$$

$$\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - + \cdots & x - \frac{x^3}{6} + \frac{x^5}{120} - + \cdots \\
 & x - \frac{x^3}{2} + \frac{x^5}{24} - + \cdots \\
\hline
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 + - \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\hline
 & \frac{2}{15}x^5 + \cdots
\end{array}$$

so $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

and $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \boxed{\frac{1}{3}}$$

FTIS: Use L'Hôpital's rule to check this.

8.8. Applications of Taylor Series

If f has a Taylor Series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

then the n^{th} partial sums are polynomial approximations.

$$T_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $R_n(x) \to 0$ as $n \to \infty$, then $T_n(x) \to f(x)$ as $n \to \infty$.

When n is large, $T_n(x) \approx f(x)$, especially near a.

We already did some examples w/ the partial sums.

Ex. In Einstein's theory of special relativity the mass of an object moving w/ velocity v is:

$$m = \frac{m_0}{\sqrt{1-(v/c)^2}}$$

where m_0 is the mass of the object at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0 c^2$$

1. Show that when v is very small, this K agrees w/ Newtonian physics: $K = \frac{1}{2} m_0 v^2$.

Sol'n. $K = mc^2 - m_0c^2 = \dfrac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} - m_0 c^2$

$$= m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right)$$

v is always less than c, so we can write $\left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2}$ as a power series (binomial series) w/ $k=-\frac{1}{2}$ (we just did this).

$$\left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{-v^2}{c^2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{v^{2n}}{c^{2n}}$$

or $\quad = 1 - \frac{1}{2}x + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \dfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \dots$

where $x = \frac{-v^2}{c^2}$. So we have

$$K = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \dots \right)$$

If v is much less than c, then $\frac{v^{2n}}{c^{2n}}$ is very small. If we omit all but the first term in the parentheses, we get

$$K \approx m_0 \cancel{c^2} \underbrace{\left(\tfrac{1}{2} v^2/\cancel{c^2}\right)}_{\text{this is } (T_1 - 1)} = \boxed{\tfrac{1}{2} m_0 v^2} \quad ☺$$

2. Use Taylor's formula to calculate the difference in these formulas when $|v| \leq 100$ m/s

Sol'n. Since we used T_1 to approximate $\left(1-\frac{v^2}{c^2}\right)^{-1/2}$, we want to find an upper bound on R_1.

Taylor's formula: $\quad R_1(x) = \dfrac{f''(z)}{2!} x^2 \qquad \left(\text{again } x = \frac{-v^2}{c^2}\right)$

In particular, put $\overset{f(x)}{\overset{\shortparallel}{K}} = m_0 c^2\left[(1-x)^{-1/2} - 1\right]$

Find $f''(x)$:

$$f'(x) = m_0 c^2 \left(-\tfrac{1}{2}\right)(1-x)^{-3/2}$$

$$f''(x) = m_0 c^2 \left(-\tfrac{1}{2}\right)\left(\tfrac{-3}{2}\right)(1-x)^{-5/2}$$

$$= m_0 c^2 \left(\tfrac{3}{4}\right)(1-x)^{-5/2}$$

Then $R_1(x) = \dfrac{3 m_0 c^2}{8(1-z)^{5/2}} \quad \dfrac{v^4}{c^4}$

where z is between 0 and $\frac{-v^2}{c^2}$. We have

$c = 3 \times 10^8$ m/s and $|v| \leq 100$ m/s

So $\left|\dfrac{v^2}{c^2}\right| \leq \dfrac{10000}{9 \times 10^{16}}$

In particular,

$$R_1(x) \leq \frac{\frac{3}{8} m_0 (9 \times 10^{16})(100/c)^4}{(1 - 100^2/c^2)^{5/2}} < \underline{(4.17 \times 10^{-10})\, m_0}$$

This is a maximum bound on the error. It's very small. So this is a good approximation!