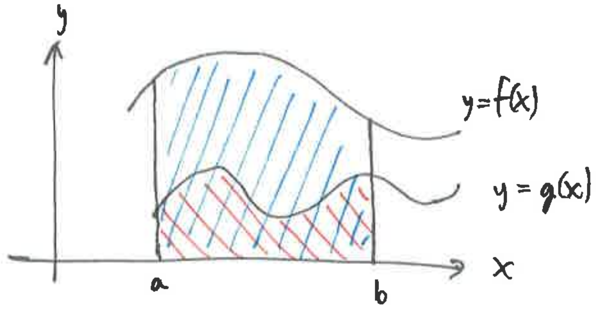
Chapter 7: Applications of Integration

7.1: Area between curves



Area between f and g = area under f − area under g.

or Area = $\int_a^b [f(x) - g(x)]\, dx$

Ex. Area between $y=e^x$ and $y=x$ from $x=0$ to $x=1$.

$y=e^x$ is on top.

$$\text{Area} = \int_0^1 e^x - x\, dx = e^x - \frac{1}{2}x^2 \Big|_0^1 = e - \frac{1}{2} - 1 + 0$$
$$= \boxed{e - 3/2}$$

Ex. Find the area enclosed by the parabolas $y=x^2$, $y=2x-x^2$

$x^2 = 2x - x^2$

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$x=0 \quad x=1$

$y=x^2$ is not on top

$$A = \int_0^1 2x^2 - 2x\, dx = \frac{2}{3}x^3 - x^2 \Big|_0^1$$
$$= \left|\frac{2}{3} - 1\right| = \left|-\frac{1}{3}\right| = \frac{1}{3}$$

Ex. $y=x-1$, $y^2 = 2x+6$ Find area enclosed.

$x = y+1 \qquad x = \frac{1}{2}y^2 - 3$

$\frac{1}{2}y^2 - 3 = y+1$

$\frac{1}{2}y^2 - y - 4 = 0$

$y^2 - 2y - 8 = 0$

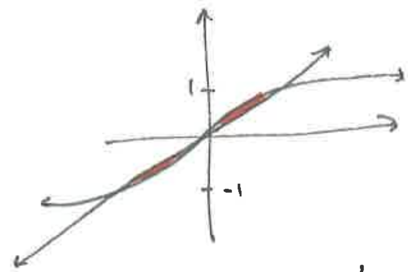
$(y-4)(y+2)$

$y=4 \quad y=-2$

$$\int_{-2}^{4} (y+1) - (\tfrac{1}{2}y^2 - 3)\, dy$$
$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy$$
$$= -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y \Big|_{-2}^{4} = \dots = \boxed{18}$$

Ex. $x=y$, $x=y^3$ find area enclosed

$$\left.\begin{aligned} y &= y^3 \\ y^3 - y &= 0 \\ (y+1)(y-1)\,y &= 0 \end{aligned}\right\} y = \pm 1, 0$$



These pieces have the same area.

So, $A = 2\int_0^1 y - y^3 \, dy$

$= 2\left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right)\Big|_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = 2\left(\frac{1}{4}\right) = \boxed{\frac{1}{2}}$

$y = \sin 2x \qquad y = \cos x \qquad 0 \to \frac{\pi}{2}$

$\sin 2x = \cos x$

$2\sin x \cos x = \cos x$

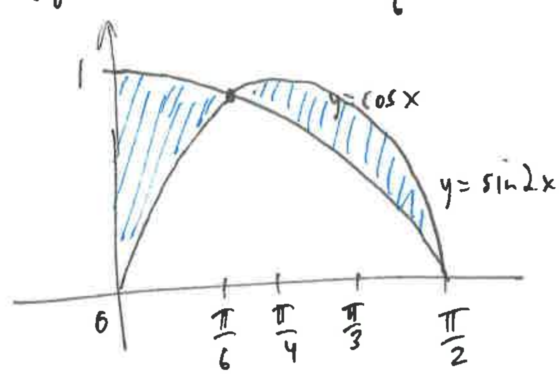
$2\sin x \cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0 \qquad \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{2} \qquad \Rightarrow x = \frac{\pi}{6}$

$$\int_0^{\pi/6} \cos x - \sin 2x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x \, dx$$



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7.1 : Volumes

Consider a cylinder:

$V = \underbrace{\pi r^2}_{\text{Area of a cross section}} h$

Idea: Slice out a cross section, Find its area, and integrate in the direction of the height.

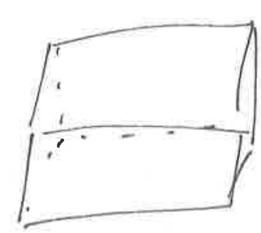
$$V = \int_a^b \text{Area } d\,\text{height}$$

A more general cylinder

slice looks like

Find area, add 'em up.

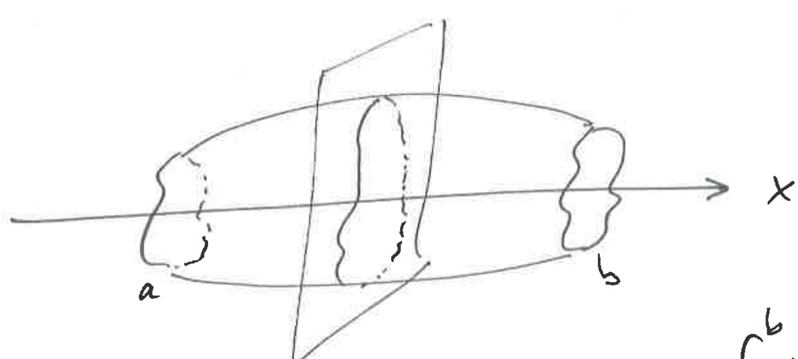
A "rectangular cylinder" (a box):



Slice 

$$V = \int_a^b (l \cdot w)\, dh$$

You can imagine doing this for more exotic shapes:

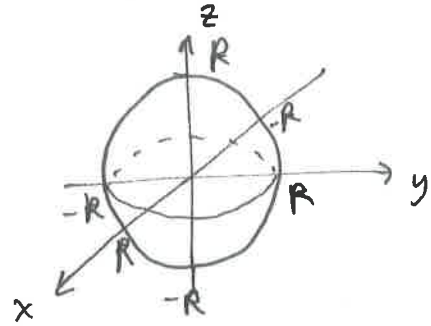


$$V = \int_a^b A(\text{slice})\, dx$$

Slice:

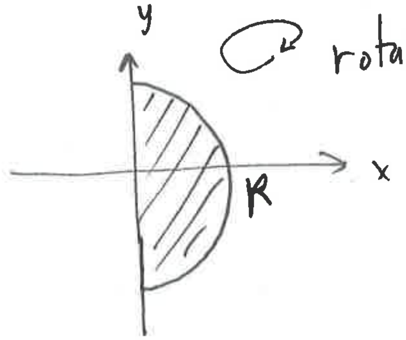
Area depends on x

Ex. Volume of a sphere (ball) w/ radius R



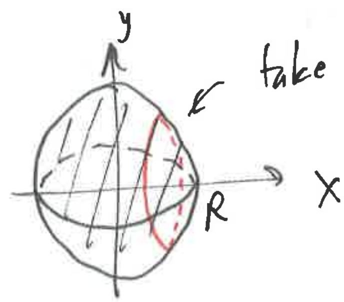
Not necessarily good to think of it this way.

Instead think:

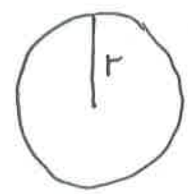


rotated around the y-axis 2π rads.

to become



take out a vertical slice:



Area $= \pi r^2$

Can we find $r = r(x)$?

Circle given by $x^2+y^2=R^2$

so $y=\sqrt{R^2-x^2}$ in QI, and $r(x)=y=\sqrt{R^2-x^2}$

So, area of slice $= \pi r^2 = \pi\left(\sqrt{R^2-x^2}\right)^2 = \pi(R^2-x^2)$

Integrate in x-direction from $-R$ to R:

$$V=\int_{-R}^{R}\pi(R^2-x^2)\,dx = 2\pi\int_0^R R^2-x^2\,dx$$

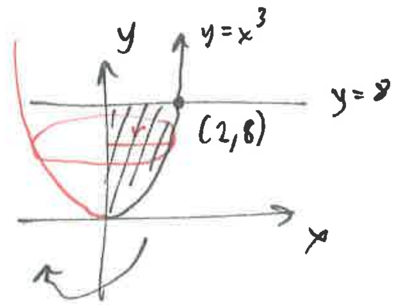
$$=2\pi\left(R^2x-\tfrac{1}{3}x^3\right)\Big|_0^R$$

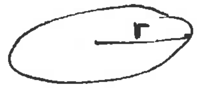
$$=2\pi\left(R^3-\tfrac{1}{3}R^3\right) = 2\pi\left(\tfrac{2}{3}R^3\right)$$

$$=\boxed{\tfrac{4}{3}\pi R^3} \; !$$

Ex. $y=x^3$, $y=8$, $x=0$

Revolve around y-axis.

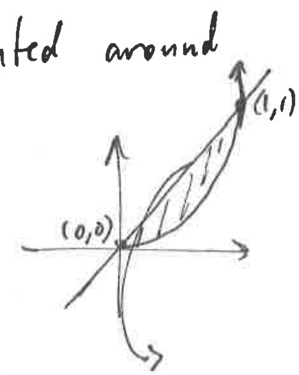


Slice horizontally: 

$$r(y)=\sqrt[3]{y}$$

$$V=\int_0^8 \pi r^2\,dy = \pi\int_0^8 y^{2/3}\,dy = \frac{3\pi}{5}\,y^{5/3}\Big|_0^8 = \frac{3\pi}{5}(32)$$
$$= \boxed{\frac{96\pi}{5}}$$

Ex. R is enclosed by $y=x$ and $y=x^2$, then rotated around x-axis.



Vertical slice: 

$$A=\pi R^2-\pi r^2=\pi(R^2-r^2)$$

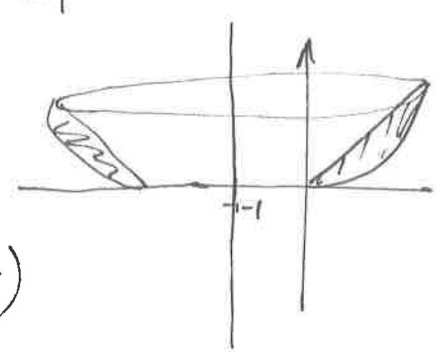
$$\left.\begin{array}{l} R(x)=x \\ r(x)=x^2 \end{array}\right\} A(x)=\pi(x^2-x^4)$$

~~Then~~ $\cancel{V=\pi\int_0^1 x-x^2\,dx = \pi\left(\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right)\Big|_0^1 = \pi\left(\tfrac{1}{2}-\tfrac{1}{3}\right)=\tfrac{\pi}{6}}$

$$V=\pi\int_0^1 x^2-x^4\,dx = \pi\left(\tfrac{1}{3}x^3-\tfrac{1}{5}x^5\right)\Big|_0^1 = \pi\left(\tfrac{1}{3}-\tfrac{1}{5}\right)=\boxed{\frac{2\pi}{15}}$$

Ex. Rotate the same R about $x=-1$

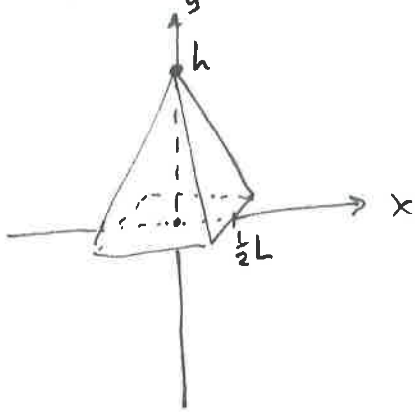
Horizontal slice: 



$$\left.\begin{array}{l} R(y)=\sqrt{y}+1 \\ r(y)=y+1 \end{array}\right\} A(y)=\pi\left((\sqrt{y}+1)^2-(y+1)^2\right)$$
$$=\pi\left(y+2\sqrt{y}+\cancel{1}-y^2-2y-\cancel{1}\right)$$
$$=\pi(-y^2-y+2\sqrt{y})$$

$$V=\pi\int_0^1 -y^2-y+2y^{1/2}\,dy = \pi\left(-\tfrac{1}{3}y^3-\tfrac{1}{2}y^2+\tfrac{4}{3}y^{3/2}\right)\Big|_0^1 = \pi\left(\tfrac{4}{3}-\tfrac{1}{3}-\tfrac{1}{2}\right)=\boxed{\frac{\pi}{2}}$$

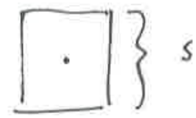
Ex. Pyramid w/ square base, side length of base $= L$, and height $= h$.

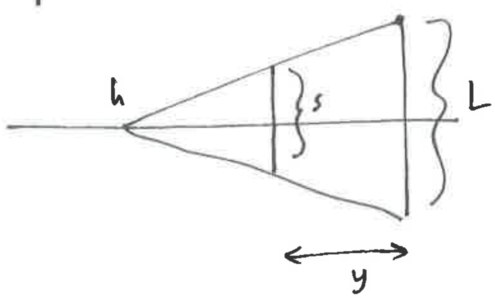


need eqn of line w/ y-int h, x-int $\frac{1}{2}L$.

$$y = -\frac{2h}{L}x + h \Rightarrow x = \frac{-L}{2h}y + \frac{L}{2}$$

Horizontal slice:





(turn it around)

By similar triangles:

$$s = 2x = \frac{-L}{h}y + L$$

Area of square is s^2

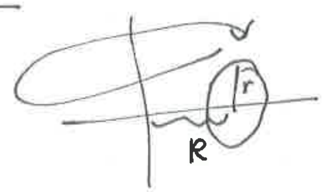
$$s^2 = \left(-\frac{L}{h}y + L\right)^2 = \frac{L^2}{h^2}y^2 - \frac{2L^2}{h}y + L^2$$

Then $V = \int_0^h \frac{L^2}{h^2}y^2 - \frac{2L^2}{h}y + L^2\,dy = \left(\frac{L^2}{3h^2}y^3 - \frac{L^2}{h}y^2 + L^2 y\right)\Big|_0^h$

$$= \frac{L^2}{3h^2}h^3 - \frac{L^2}{h}h^2 + L^2 h$$

$$= \frac{1}{3}L^2h - L^2h + L^2h = \boxed{\frac{1}{3}L^2h} \quad ! \; \ddot{\smile}$$

HW. Volume of a ~~[illegible]~~ torus.



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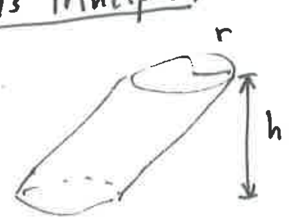
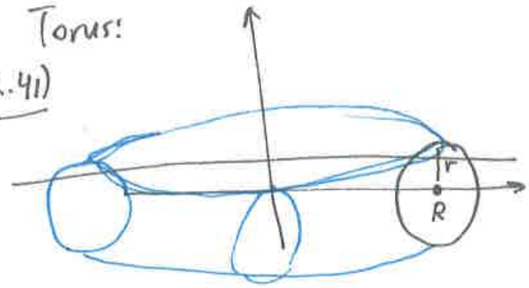
Cavalieri's Principle:



$V = \pi r^2 h$

7.3. Volume by Cylindrical Shells

Torus:
7.2.41)



Horizontal slice

Equation of circle $(x-R)^2+y^2=r^2$

Integrate in y-direction: $(x-R)^2 = r^2-y^2$

$$x-R = \pm\sqrt{r^2-y^2}$$

$$x = R\pm\sqrt{r^2-y^2}$$

outer radius: $R+\sqrt{r^2-y^2}$

inner radius: $R-\sqrt{r^2-y^2}$

Area: $\pi\left[\left(R+\sqrt{r^2-y^2}\right)^2-\left(R-\sqrt{r^2-y^2}\right)^2\right]$

$$=\pi\left[\cancel{R^2}+2R\sqrt{r^2-y^2}+\cancel{r^2}-\cancel{y^2}-\cancel{R^2}+2R\sqrt{r^2-y^2}-\cancel{r^2}+\cancel{y^2}\right]$$

$$=4\pi R\sqrt{r^2-y^2}$$

Volume: $4\pi R\int_{-r}^{r}\sqrt{r^2-y^2}\,dy = 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$

let $y=r\sin\theta$ $\quad\theta(0)=0$

$dy=r\cos\theta\,d\theta$ $\quad\theta(r)=\frac{\pi}{2}$

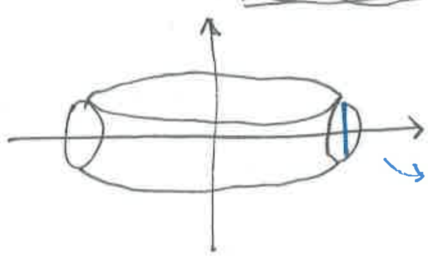
$$=8\pi R\int_0^{\pi/2} r^2\cos^2\theta\,d\theta$$

$$=8\pi Rr^2\cdot\frac{1}{2}\int_0^{\pi/2}1+\cos 2\theta\,d\theta$$

$$=4\pi Rr^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$=\boxed{2\pi^2Rr^2}$$

Another way:



Take a vertical slice, and rotate it around. Get a cylindrical shell.

Area $=2\pi rh$

$r=x$

$h=2\sqrt{r^2-(x-R)^2}$

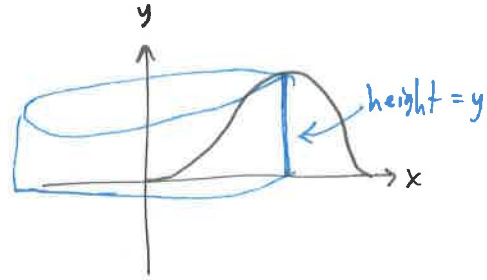
$$V=4\pi\int_{R-r}^{R+r}x\sqrt{r^2-(x-R)^2}\,dx$$

$\underline{\text{Ex}}$. $y = 2x^2 - x^3$, $y = 0$ rotate around y-axis.

Find intersections: $2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x = 0 \quad x = 2$

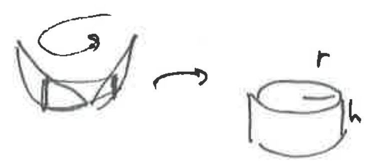


$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx = 2\pi \int_0^2 2x^3 - x^4\,dx$$

$$= 2\pi \left(\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right)\Big|_0^2 = 2\pi\left(\tfrac{16}{2} - \tfrac{32}{5}\right)$$

$$= 2\pi\left(\frac{80-64}{10}\right) = \boxed{\frac{16\pi}{5}}$$

$\underline{\text{Ex}}$. $y = x$, $y = x^2$ rotate around y-axis.



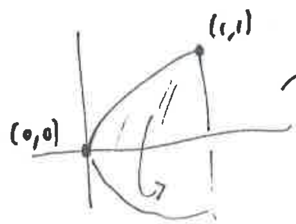
$r = x \quad h = \cancel{y} - x^2$

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 x^2 - x^3\,dx$$

$$= 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1$$

$$= 2\pi\left(\frac{4-3}{12}\right) = \boxed{\frac{\pi}{6}}$$

$\underline{\text{Ex}}$. $y = \sqrt{x}$ from $x = 0$ to 1 rotated around x-axis.



shell:

$h = 1 - y^2$

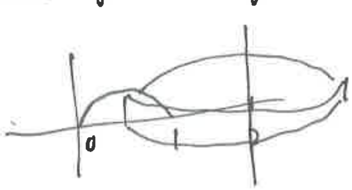
$r = y$

$$V = \int_0^1 2\pi y(1 - y^2)\,dy$$

$$= 2\pi \int_0^1 y - y^3\,dy$$

$$= 2\pi\left(\tfrac{1}{2}y^2 - \tfrac{1}{4}y^4\right)\Big|_0^1 = 2\pi\left(\tfrac{1}{4}\right) = \boxed{\pi/2}$$

$\underline{\text{Ex}}$. $y = x - x^2$ $y = 0$ around line $x = 2$



$$\left.\begin{array}{l} r = 2 - x \\ h = x - x^2 \end{array}\right\} \quad V = \int_0^1 2\pi(2-x)(x-x^2)\,dx = 2\pi \int_0^1 2x - x^2 - 2x^2 + x^3\,dx$$

$$= 2\pi\left[x^2 - x^3 + \tfrac{1}{4}x^4\right]_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

Ex. Right Circular Cone w/ height h and base radius r.

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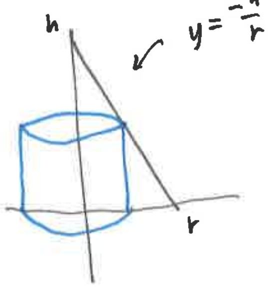

$y = -\frac{h}{r}x + h$

$$V = 2\pi \int_0^r x\left(-\frac{h}{r}x + h\right) dx$$

$$= 2\pi \int_0^r -\frac{h}{r}x^2 + hx \, dx$$

$$= 2\pi \left(-\frac{h}{3r}x^3 + \frac{h}{2}x^2 \right)\Big|_0^r = 2\pi \left(-\frac{1}{3}hr^2 + \frac{1}{2}hr^2 \right)$$

$$= \frac{2\pi}{6}hr^2 = \boxed{\frac{\pi}{3}hr^2}$$

Back to torus:

$$V = 4\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2} \, dx \qquad \begin{array}{l} u = x - R \\ du = dx \end{array} \qquad \begin{array}{l} u(R-r) = -r \\ u(R+r) = +r \end{array} \qquad x = u + R$$

$$= 4\pi \int_{-r}^{r} (u+R)\sqrt{r^2 - u^2} \, du = 4\pi \int_{-r}^{r} u\sqrt{r^2-u^2} \, du + 4\pi \int_{-r}^{r} R\sqrt{r^2-u^2} \, du$$

$$= \ldots$$

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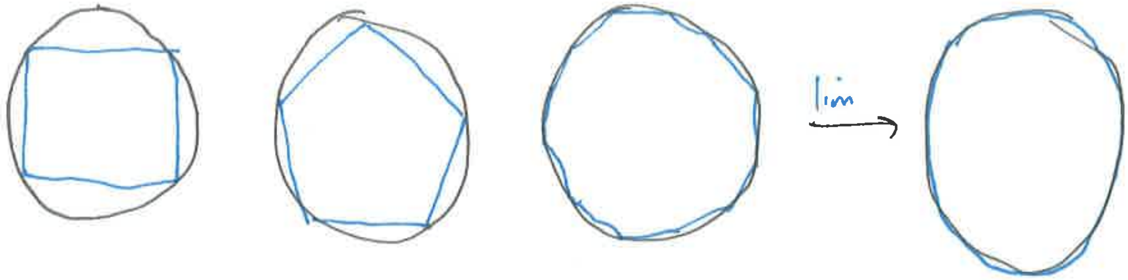
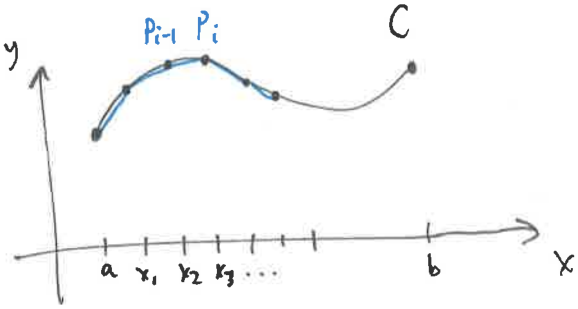
7.4. Arc Length

Idea. Circumference of a circle:



For a ~~[illegible]~~ curve:



1. Partition $[a,b]$ just as you would for a Riemann sum.
2. "Connect the dots" with line segments.
3. The length of the segments added up is approximately the length of C.
4. Take a limit to make it exact.

$$|P_{i-1}P_i| = \sqrt{(x_{i-1}-x_i)^2 + (y_{i-1}-y_i)^2}$$
$$= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|}$$

For a uniform partition Δx_i is constant $= \Delta x$.

By the Mean Value Theorem (for derivatives), there is a x_i^* in $[x_{i-1}, x_i]$ such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_{i-1} - x_i)$

OR $\quad \Delta y_i = f'(x_i^*)\,\Delta x$

So,
$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$
$$= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2}$$
$$= \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

and
$$L = \lim_{n\to\infty} \sum_{n=1}^{\infty} \sqrt{1+f'(x_i^*)^2}\,\Delta x = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \boxed{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

Ex. Find the arc length of the curve $y^2 = x^3$ between $(1,1)$ and $(4,8)$.

$$y = x^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$$

$$L = \int_0^2 \sqrt{1 + \tfrac{9}{4}x}\, dx \qquad \begin{array}{l} u = 1 + 9/4\, x \\ du = 9/4\, dx \end{array} \qquad \begin{array}{l} u(0) = 1 \\ u(2) = \frac{22}{4} \end{array}$$

$$= \frac{4}{9}\int_1^{\frac{22}{4}} \sqrt{u}\, du = \frac{4}{9}\cdot\frac{2}{3}\, u^{3/2}\Big|_1^{22/4} = \frac{8}{27}\left(\left(\frac{22}{4}\right)^{3/2} - 1\right) \approx 3.5255$$

Exercise. Show that $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

if C is given by an invertible function between $P(a,c)$ and $Q(b,d)$.

Ex. Find the arc length of $x = y^2$ from $(0,0)$ to $(1,1)$.
or $y = \sqrt{x}$

$$\left.\begin{array}{l} x = y^2 \\ \frac{dx}{dy} = 2y \\ \left(\frac{dx}{dy}\right)^2 = 4y^2 \end{array}\right\} \quad L = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

let $y = \frac{1}{2}\tan\theta \Rightarrow \theta = \arctan(2y)$
$dy = \frac{1}{2}\sec^2\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\text{atan}(2)} \sec^3\theta\, d\theta = \cancel{\frac{1}{2}\int_0^{\text{atan}(2)} (1 + \tan^2\theta)\sec\theta\, d\theta}$$

$$\cancel{= \frac{1}{2}\int_0^{\text{atan}(2)} \sec\theta\, d\theta + \frac{1}{2}\int_0^{\text{atan}(2)} \tan^2\theta\sec\theta\, d\theta}$$

$$\frac{1}{2}\int_0^{\operatorname{atan}(2)} \sec^3\theta \, d\theta$$

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$$\int \sec^3\theta \, d\theta$$

$u = \sec\theta$ $\qquad dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta$ $\qquad v = \tan\theta$

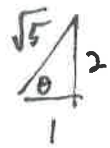
$$= \sec\theta\tan\theta - \int \sec\theta \overbrace{\tan^2\theta}^{\sec^2\theta - 1} \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$\Rightarrow \quad 2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$\theta = \arctan(2)$ means $\tan\theta = \frac{2}{1}$



$\tan 0 = 0$

$\sec\theta = \sqrt{5}$

$\sec 0 = 1$

So, $$L = \frac{1}{2}\int_0^{\arctan(2)} \sec^3\theta \, d\theta = \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| \Big|_0^{\operatorname{atan}(2)}$$

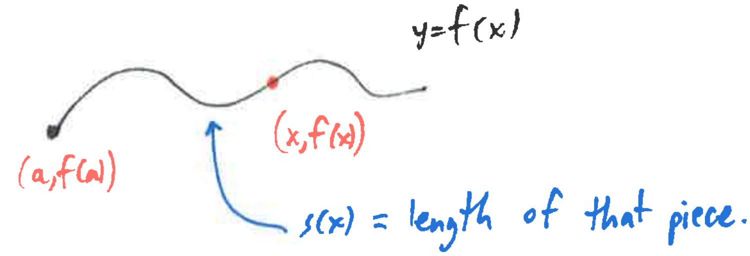
$$= \frac{1}{4}\sqrt{5}(2) + \frac{1}{4}\ln|\sqrt{5} + 2|$$

$$= \boxed{\frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}} \approx 1.4789$$

The arc length function

$$s(x) = s(f,x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

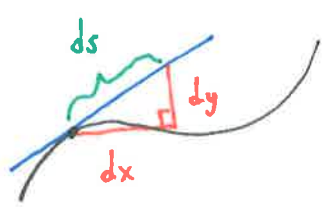
measures the length of a curve $y=f(x)$ starting at a.



By the fundamental theorem of calculus:

$$\frac{ds}{dx} = \frac{d}{dx}\int_a^x \sqrt{1+[f'(t)]^2}\, dt = \sqrt{1+[f'(x)]^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

so $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ or $ds^2 = dy^2 + dx^2$



so ds is the change in the tangent direction of the curve. It's called a <u>length element</u>.

We can write: $L = \int ds$ to get a nice short form of the arc length formula.

Ex. let $f(x) = x^2 - \frac{1}{8}\ln(x)$ and $a=1$ the starting point.
Find the arc length function.

$$f'(x) = 2x - \frac{1}{8x}$$

$$f'(x)^2 = 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$1 + f'(x)^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1+f'(x)^2} = \sqrt{\left(2x+\frac{1}{8x}\right)^2} = 2x + \frac{1}{8x}$$

$$s(x) = \int_{a=1}^{x} 2t + \frac{1}{8t}\,dt = t^2 + \frac{1}{8}\ln t \Big|_{a=1}^{x} = \boxed{x^2 + \frac{1}{8}\ln x - 1}$$

Now, find the length of the curve from 1 to e

$$s(e) = e^2 + \frac{1}{8}\ln e - 1 = \boxed{e^2 - 1}$$

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373.$$

7.5. Applications to Physics

Work:

Newton's 2nd Law of motion: $F = ma$ or $F = m\dfrac{d^2 s}{dt^2}$

where $s(t)$ is the position function of a particle.

Work = Force × distance

$$W = Fd$$

This works great as long as the force is constant. But if force is a function, then

$$\boxed{W = \int_a^b f(x)\,dx}$$

is the work done by f between $x=a$ and $x=b$.

Ex. When a particle is located x units from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x=1$ to $x=3$?

$$W = \int_1^3 x^2 + 2x\,dx = \frac{1}{3}x^3 + x^2 \Big|_1^3 = \frac{27}{3} + 9 - \frac{1}{3} - 1$$

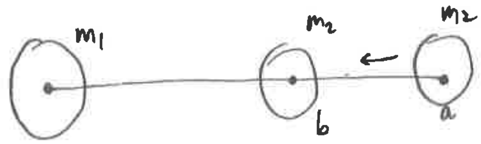
$$= 27 - \frac{4}{3} = \frac{54-4}{3} = \boxed{\frac{50}{3}} \text{ ft-lb.}$$

Ex. (21) Newton's law of gravitation: two bodies w/ masses m_1 and m_2 attract each other w/ a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the ~~centers~~ centers and G is the gravitational constant.

If one body is fixed, find the work done to move the other from $r=a$ to $r=b$



$$W = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1m_2\int_a^b r^{-2}\,dr$$

$$= Gm_1m_2\left(-\frac{1}{r}\right)\Big|_b^a = \left|Gm_1m_2\frac{1}{b} - Gm_1m_2\frac{1}{a}\right|$$

Compute the work required to launch a 1000 kg satellite vertically to an orbit 1000 km high.

Take $m_E = 5.98 \times 10^{24}$ kg

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$ and

$r_E = 6.37 \times 10^6$ m

so $b = 6.37 \times 10^6$ and $a = 6.37\times 10^6 + 10^6 = 7.37 \times 10^6$

Plug and chug to get:

$$\approx \boxed{8.496 \times 10^9 \text{ Joules !}}$$

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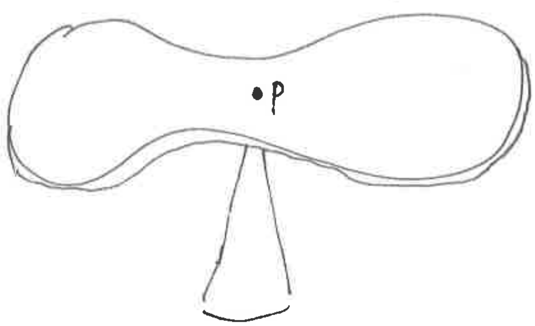
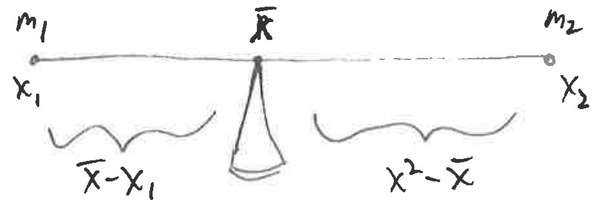
Moments and Centers of Mass:



P = the center of mass (gravity)

If this shape were a thin plate, it would balance perfectly at this point.

First, look at this problem as a 1-d problem:



Two masses are attached to opposite ends of a rod. We do we need to put the fulcrum so that the rod will balance?

Archimedes discovered the "Law of the lever":

$$m_1 d_1 = m_2 d_2$$

$$m_1(\bar{x}-x_1) = m_2(x_2-\bar{x}) \qquad \text{solve for } \bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\boxed{\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}}$$

In general, if a system has n particles:

$$\boxed{\bar{x} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i}, \quad \text{where } m = \sum_{i=1}^{n} m_i$$

The quantity $\boxed{M := \sum_{i=1}^{n} m_i x_i}$ is called the moment of the system about the origin.

In a 2D system w/ particles at $(x_1,y_1) \ldots (x_n,y_n)$, there are moments

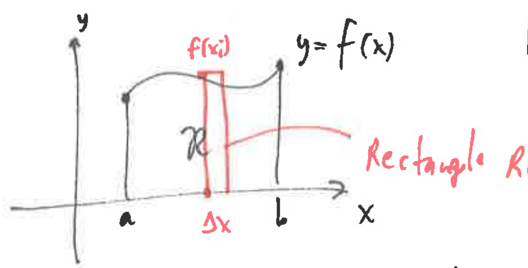
$$\begin{cases} M_y = \sum_{i=1}^{n} m_i x_i & \text{and} \\ M_x = \sum_{i=1}^{n} m_i y_i \end{cases}$$

M_y measures tendency to rotate about y-axis and M_x around x-axis.

The center of mass of a 2D system is given by

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \quad \text{where } m = \sum m_i .$$

As usual, when we switch from discrete to continuous data, the sums become an integrals.



Let $\mathcal{R}$ be a flat plate w/ uniform density ρ.

For any rectangle, Area $= f(x_i)\Delta x$ so its mass is $\rho f(x_i)\Delta x$

The moment of R_i about y-axis is $M_y(R_i) = [\rho f(x_i)\Delta x]\underbrace{\bar{x}_i}_{\text{distance to y-axis}}$

As an end result:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(x_i)\Delta x = \boxed{\rho \int_a^b x f(x)\,dx .}$$

For M_x

$c = (\bar{x}_i, \frac{1}{2} f(x_i))$

$$M_x(R_i) = [\rho f(\bar{x}_i)\Delta x]\underbrace{\tfrac{1}{2} f(x_i)}_{\text{distance to x-axis.}}$$

Then $M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \boxed{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}$

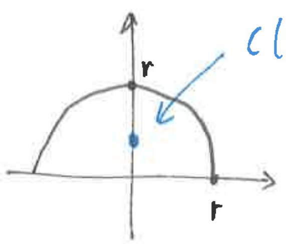
As before:

$$\bar{x} = \frac{M_y}{m} = \frac{\rho\int_a^b x f(x)\,dx}{\rho\int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} = \boxed{\frac{1}{A}\int_a^b x f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho\int_a^b f(x)\,dx} = \boxed{\frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx}$$

* center of mass is independent of the density (as long as density is constant).

Ex. Find the center of mass of A semi-circular plate.



$c(\bar{x},\bar{y})$ is somewhere around here.

$f(x) = \sqrt{r^2 - x^2}$ $a=-r,\ b=r$

$A = \frac{1}{2}\pi r^2$ by geometry.

So, $\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx = \frac{2}{\pi r^2}\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx$

$u = r^2 - x^2$
$du = -2x\,dx$

$u(r) = 0$!
$u(-r) = 0$!

since u is even!

$$= -\frac{2}{\pi r^2}\int_0^0 \sqrt{u}\,du = \frac{-2}{\pi r^2}\left[\frac{2}{3}u^{3/2}\right]_0^0 = 0$$

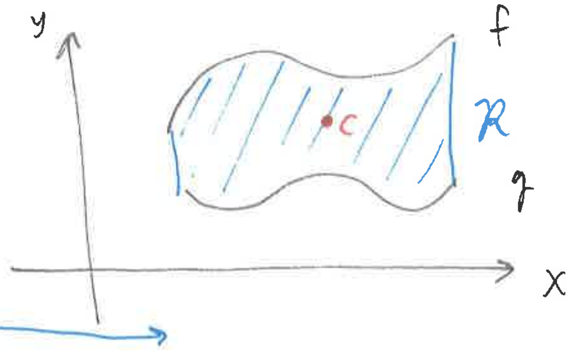
OR, by symmetry principle: center of mass must lie on axis of symmetry.

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\pi r^2}\int_{-r}^{r} r^2 - x^2\,dx = \frac{2}{\pi r^2}\int_0^r r^2 - x^2\,dx$$

$$= \frac{2}{\pi r^2}\left[r^2x - \frac{1}{3}x^3\right]_0^r = \frac{2}{\pi r^2}\left[r^3 - \frac{1}{3}r^3\right] = \frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \boxed{\frac{4r}{3\pi}}$$

So, center of mass is at $\boxed{(\bar{x},\bar{y}) = \left(0, \frac{4r}{3\pi}\right)}$

What about a region like this:



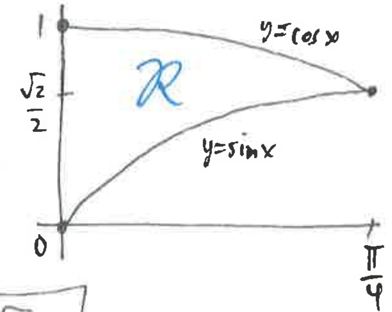
"centroid" of $\mathcal{R}$ is given by

$$\begin{cases} \bar{x} = \frac{1}{A}\int_a^b x\left[f(x)-g(x)\right]dx \\ \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx \end{cases}$$

Ex in text: $y=x$, $y=x^2$, $x=0$, $x=1$

Ans: $(\bar{x},\bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$

Ex. (41) $y=\sin x$ $\quad y=\cos x \quad x=0, \ x=\frac{\pi}{4}$



$$A = \int_0^{\pi/4} \cos x - \sin x \, dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} = \frac{2\sqrt{2}}{2} - 1 = \boxed{\sqrt{2}-1}$$

$$\bar{x} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\cos x - x\sin x \, dx = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\cos x \, dx - \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\sin x \, dx$$

$$= \frac{1}{\sqrt{2}-1}\left[x\sin x + \cos x - \left(-x\cos x + \sin x\right)\right]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}-1}\left[x\sin x + x\cos x + \cos x - \sin x\right]_0^{\pi/4} = \frac{\sqrt{2}\,\pi - 4}{4(\sqrt{2}-1)} \approx 0.267$$

$$\bar{y} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \frac{1}{2}(\cos x - \sin x)^2 \, dx = \frac{1}{2(\sqrt{2}-1)}\int_0^{\frac{\pi}{4}} \cos^2 x - 2\sin x\cos x + \sin^2 x \, dx$$

$$= \frac{1}{2(\sqrt{2}-1)}\int_0^{\frac{\pi}{4}} 1 - 2\sin x\cos x \, dx = \frac{1}{2(\sqrt{2}-1)}\left[x + \cos^2 x\right]_0^{\pi/4}$$

$$= \frac{1}{2(\sqrt{2}-1)}\left[\frac{\pi}{4} + \frac{1}{2} - 1\right] = \frac{\pi - 2}{8(\sqrt{2}-1)} \approx 0.345$$

So, $\boxed{c = (\bar{x},\bar{y}) = \left(\frac{\sqrt{2}\,\pi - 4}{4(\sqrt{2}-1)}, \frac{\pi-2}{8(\sqrt{2}-1)}\right) \approx (0.267, 0.345)}$

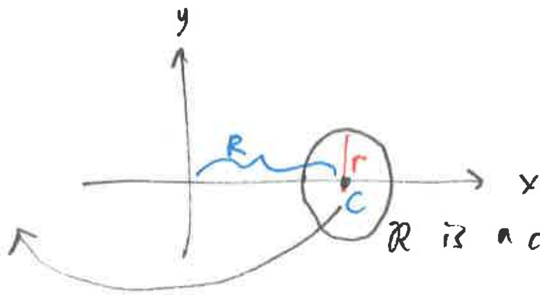
This leads to a remarkable theorem:

<u>The Theorem of</u> $\boxed{\text{Pappus}}$:

Greek mathematician, lived in 300s AD (CE).

Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line ℓ in the same plane. If $\mathcal{R}$ is rotated about ℓ, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$ (i.e., the circumference of the circle created by rotating the centroid).

We've already used this theorem! The torus:



$\mathcal{R}$ is a circle of radius r

Volume = "$A(\mathcal{R})$" $\times$ "$d(C)$"

$A(\mathcal{R}) = \pi r^2$

$d(C) = 2\pi R$

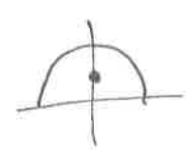
So, $\boxed{V = 2\pi^2 r^2 R}$

Proof of this is in the book, but we already justified it using Cavalieri's principle (loosely).

(48)

Ex. Use Pappus' Theorem to find the volume of a sphere of radius r.

We've already found that $C = \left(0, \frac{4r}{3\pi}\right)$ for



And area of semi-circle is $\frac{1}{2}\pi r^2$. So

$$V = \frac{1}{\cancel{2}}\pi r^2 \left(\cancel{2}\cancel{\pi} \cdot \frac{4r}{3\cancel{\pi}}\right) = \boxed{\frac{4}{3}\pi r^3} \quad !$$

(49) is a "good problem".

7.6. Differential Equations

A <u>differential equation</u> is an equation involving an unknown function and one or more of its derivatives.

Ex. $y' = xy$ We want to solve for y.

$$\frac{dy}{dx} = xy$$

$$\int \frac{dy}{y} = \int x \, dx$$

$$\ln y = \frac{1}{2}x^2 + C$$

$$y = e^{\frac{1}{2}x^2 + C} = e^{C} e^{\frac{1}{2}x^2} = K e^{\frac{1}{2}x^2}$$ where K is an unknown constant.

An equation like this is called <u>separable</u>, because we can separate x and y.

In general, separable equations look like:

$$\frac{dy}{dx} = f(x)\, g(y)$$

so, we can solve $\dfrac{dy}{g(y)} = f(x)\, dx$

A better (or easier) situation:

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$ so that

$$h(y)\, dy = f(x)\, dx.$$

Ex. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2} \Rightarrow y^2 dy = x^2 dx$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$y^3 = x^3 + C$$

$$\boxed{y = \sqrt[3]{x^3 + C}}$$

If we want to know C, then we need an initial condition.

Suppose ~~y(0)~~ we want (or know) that $y(0) = 2$. Then

$$y(0) = \sqrt[3]{0^3 + C} = \sqrt[3]{C} = 2 \Rightarrow C = 2^3 = 8$$

and $y = \sqrt[3]{x^3 + 8}$ is the particular solution to this problem.

Ex. The DE $y' = ky$ occurs in nature all of the time. It says that the growth (y') of a quantity (y) is proportional to the ~~popul~~ quantity itself. e.g. pop'n growth, interest, radioactive decay.

Let's solve it. Assume $y = y(t)$.

$$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt$$

$$\Rightarrow \ln y = kt + C$$
$$e^{\ln y} = e^{kt + C}$$

$$y = e^{C} e^{kt} = Ce^{kt}$$

$$\boxed{y = Ce^{kt}}$$

Eg. $\begin{cases} \text{Pop'n:} & P = P_0 e^{kt} \\ \text{Int:} & A = Pe^{rt} \end{cases}$

Recall that $\frac{dy}{dx}$ is the slope of the tangent line of a function. So $\frac{dy}{dx} = F(x,y)$ tells us what the slope ~~of a function~~ ~~y~~ looks like at a point (x,y).

We can graph these slopes as small line segments. Then a solution to a DE is obtained by choosing a starting point and following the slopes to draw a graph. These graphs are called slope fields, direction fields, flow fields, etc.

Ex. (10) $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$ $\Rightarrow$ $\displaystyle\int \frac{1+y^2}{y}\,dy = \int \cos x\,dx$

$y(0) = 1$

$$\ln y + \tfrac{1}{2}y^2 = \sin x + C$$

$$\cancel{\ln 1} + \tfrac{1}{2}1^2 = \cancel{\sin 0} + C$$

$$C = \tfrac{1}{2}$$

So, sol'n is $\boxed{\ln y + \tfrac{1}{2}y^2 = \sin x + \tfrac{1}{2}}$

Can't simplify this much more.

Ex. (7) $\dfrac{du}{dt} = 2t + 2u + t + tu$

$= \cancel{\cos}\, 2(1+u) + t(1+u)$

$= (2+t)(1+u)$

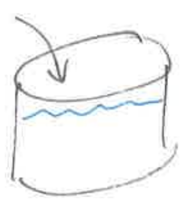
$\Rightarrow$ $\dfrac{du}{1+u} = (2+t)\,dt$

$$\ln|1+u| = 2t + \tfrac{1}{2}t^2 + C$$

$$1+u = Ce^{2t+\frac{1}{2}t^2}$$

$$\boxed{u(t) = Ce^{2t+\frac{1}{2}t^2} - 1}$$

Mixing Problems



5000 L of water
w/ 20 kg of dissolved salt

In: water enters tank at 25 L/min
w/ 0.03 kg/L of salt dissolved

Out: Thoroughly mixed solution is drained at same rate.

Q: Find how much salt is in tank after half an hour.

~~doesn't look good.~~

---

let $y(t)$ = amt of salt at time t

We know $y(0) = 20$. We want $y(30)$.

We know $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$ in kg/min

rate in: $\left(0.03\ \text{kg}/\text{L}\right)\left(25\ \text{L}/\text{min}\right) = \frac{3 \cdot 25}{100}\ \text{kg}/\text{min} = .75\ \text{kg}/\text{min}$

rate out: Conc. at time t is $\frac{y(t)\ \text{kg}}{5000\ \text{L}}$ since tank always has 5000L.

so, $\left(\frac{y}{5000}\ \frac{\text{kg}}{\text{L}}\right)\left(25\ \text{L}/\text{min}\right) = \frac{y}{200}\ \text{kg}/\text{min}$

Thus, our DE is

$$\frac{dy}{dt} = .75 - \frac{y}{200} = \frac{150 - y}{200}$$

$\boxed{y(30) \approx 38.1\ \text{kg}}$

$$\frac{dy}{150 - y} = \frac{1}{200}\, dt$$

$$-\ln(150 - y) = \frac{t}{200} + C$$

$y(0) = 20$ gives $C = -\ln(130)$

So $-\ln(150-y) = \frac{t}{200} - \ln(130)$

$$\frac{1}{150 - y} = \frac{1}{130} e^{t/200}$$

$$150 - y = \frac{130}{e^{t/200}}$$

$$\boxed{y = 150 - 130 e^{-t/200}}$$