Chapter 6: Integration Techniques

6.1: "The Amazing" Integration by Parts

We don't know how to integrate this: $\int_a^b f(x)\, g(x)\, dx$.

Let's find out how...

Let $f(x)$ and $g(x)$ be differentiable functions

Recall the product rule:

$$\frac{d}{dx}\left[f(x)\, g(x)\right] = \frac{d}{dx}\left[f(x)\right] g(x) + f(x) \frac{d}{dx}\left[g(x)\right]$$

Integrate it all wrt x:

$$\int \frac{d}{dx}\left[f(x)\, g(x)\right] dx = \int \frac{d}{dx}\left[f(x)\right] g(x)\, dx + \int f(x) \frac{d}{dx}\left[g(x)\right] dx$$

LHS: F.T.C. says $\int \frac{d}{dx}\left[\sim\right] dx = \sim$

$$f(x)\, g(x) = \int \frac{d}{dx}\left[f(x)\right] g(x)\, dx + \int f(x) \frac{d}{dx}\left[g(x)\right] dx$$

Rearrange:

$$\boxed{\int \frac{d}{dx}\left[f(x)\right] g(x)\, dx = f(x)\, g(x) - \int f(x) \frac{d}{dx}\left[g(x)\right] dx}$$

This is the integration by parts formula.

It says that we can "move the derivative from f to g".

The idea is to "get rid of g" by taking its deriv.

Ex. $\int x \sin x\, dx = -x\cos x - \int -\cos x\,(1)\, dx$

$$\left.\begin{matrix} \frac{d}{dx}\left[f(x)\right] = \sin x & g(x) = x \\ f(x) = -\cos x & \frac{d}{dx}\left[g(x)\right] = 1 \end{matrix}\right\}$$

$= -x\cos x + \int \cos x\, dx$

$= \boxed{-x\cos x + \sin x + C}$

Ex. $\int \ln x \, dx = \int \ln x \cdot 1 \, dx \quad = \quad x\ln(x) - \int x \cdot \frac{1}{x} \, dx$

$$= x\ln x - \int 1 \, dx$$

$$= \boxed{x\ln x - x + C}$$

$\frac{d}{dx}[f(x)] = 1 \qquad g(x) = \ln(x)$

$f(x) = x \qquad \frac{d}{dx}[g(x)] = \frac{1}{x}$

Ex. $\int t^2 e^t \, dt$

$\frac{d}{dt}[f(t)] = e^t \qquad g(t) = t^2$

$f(t) = e^t \qquad \frac{d}{dt}[g(t)] = 2t$

$$= t^2e^t - \underbrace{\int 2te^t \, dt}$$

$\frac{d}{dt}[f(t)] = e^t \qquad g(t) = 2t$

$f(t) = e^t \qquad \frac{d}{dt}[g(t)] = 2$

$$= t^2e^t - 2te^t + \int 2e^t \, dt$$

$$= \boxed{t^2e^t - 2te^t + 2e^t + C}$$

(*) Make a chart.

Ex. $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$\Rightarrow 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$

$\rightarrow \int e^x \sin x \, dx = \boxed{\frac{1}{2}e^x(\sin x - \cos x) + C}$

A few more Int. by Parts examples.

First:

let $u = g(x)$ $\quad dv = \frac{d}{dx}\left(f(x)\right)dx$

$du = \frac{d}{dx}\left[g(x)\right]dx \quad v = f(x)$

$\Rightarrow \int u\,dv = uv - \int v\,du$

Ex. $\int \theta \sec^2\theta \, d\theta$

$u = \theta \qquad dv = \sec^2\theta\, d\theta$

$du = 1\,d\theta \qquad v = \tan\theta$

$$\Rightarrow \int \theta \sec^2\theta = \theta\tan\theta - \int \tan\theta \, d\theta$$

$$= \theta\tan\theta - \int \frac{\sin\theta}{\cos\theta}\, d\theta$$

$y = \cos\theta$

$dy = -\sin\theta \, d\theta$

$$= \theta\tan\theta + \int \frac{1}{y}\, dy$$

$$= \theta\tan\theta + \ln|y| + C$$

$$= \theta\tan\theta + \ln|\cos\theta| + C$$

$$= \boxed{\theta\tan\theta - \ln|\sec\theta| + C}$$

How to choose u:

L log

I inv. trig.

A alg. (polynom)

T trig

E exponential

Ex. A trick:

$\int x^4 e^x \, dx$

	u	dv
+	x^4	e^x
−	$4x^3$	e^x
+	$12x^2$	e^x
−	$24x$	e^x
	24	e^x
−	0	e^x

$$e^x\left[x^4 - 4x^3 + 12x^2 - 24x + 24\right] + C$$

Ex. $\int 5x^3 \sin(5x)\, dx$

	u	dv
+	$5x^3$	$\sin 5x$
−	$15x^2$	$-\frac{1}{5}\cos 5x$
+	$30x$	$-\frac{1}{25}\sin 5x$
−	30	$\frac{1}{125}\cos 5x$
	0	$\frac{1}{525}\sin 5x$

$+C$

6.2. Trig Integrals

Ex. $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$

$= \int (1-\sin^2 x) \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$= \int 1 - u^2 \, du$

$= u - \frac{1}{3}u^3 + C$

$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

Ex. $\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx$

$u = \cos x \quad du = -\sin x$

$= -\int (1-u^2)^2 u^2 \, du$

$= -\int u^2(1-2u^2+u^4) \, du$

$= -\int u^2 - 2u^4 + u^6 \, du$

$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$

$= \boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$

Ex. $\int_0^\pi \sin^2 x \, dx$ $\qquad \sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$

$= \frac{1}{2}\int_0^\pi (1-\cos 2x) \, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x \Big]_0^\pi = \frac{1}{2}\pi - 0 = \boxed{\frac{1}{2}\pi}$

Ex. $\int \sin^4 x\, dx$

Ex. $\int \tan^6 x \sec^4 x\, dx$ $\qquad \sec^2 x\, dx$

Ex. $\int \tan^5\theta \sec^7\theta\, d\theta$ $\qquad \tan\theta\sec\theta\, d\theta$

recall: $\boxed{\int \sec x\, dx = \ln|\sec x + \tan x| + C}$

trick: mult by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$

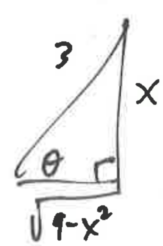
Main idea: these suck, but we can do them.
Leads to trig substitutions.

$\sqrt{a^2 - x^2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$
let $x = a\begin{Bmatrix}\sin\\ \tan\\ \sec\end{Bmatrix}\theta$	

Ex. $\int \frac{\sqrt{9-x^2}}{x^2}\, dx$

let $x = 3\sin\theta \rightarrow \sin\theta = \frac{x}{3}$
$dx = 3\cos\theta\, d\theta$

$\mapsto \int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} 3\cos\theta\, d\theta$



$= \int \frac{3\cos\theta \cdot 3\cos\theta}{9\sin^2\theta}\, d\theta$

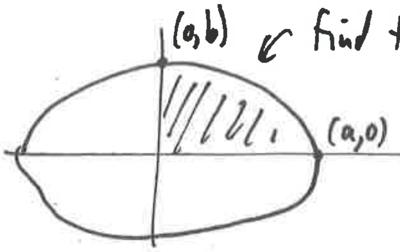
$= \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C$

$= \cancel{\ln|\sec\theta| + C} = \cancel{\ln\left|\frac{3}{\sqrt{9-x^2}}\right| + C}$ $\qquad = \boxed{-\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C}$

~~arcsin~~

8/31/12

Ex. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



← find this & mult. by 4.

using +:

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

let $x = a\sin\theta$
$dx = a\cos\theta\,d\theta$

$$= \frac{b}{a}\int_{\cancel{a}\,0}^{\cancel{a}\,\pi/2} \sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$
$\theta(0) = \sin^{-1}(0) = 0$
$\theta(a) = \sin^{-1}(1) = \frac{\pi}{2}$

$$= \frac{b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta$$

$$= \frac{ba}{2}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta$$

$$= \frac{ba}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{ba}{2}\cdot\frac{\pi}{2} - 0 = \frac{ab\pi}{4}$$

$$\Rightarrow A = \cancel{4}\left(\cancel{\tfrac{1}{4}}\right)ab\pi = \boxed{ab\pi}$$

→ First **Ex.** (60) $\int \frac{dt}{\sqrt{t^2-6t+13}}$ evaluate by first completing the square.

$$= \int \frac{dt}{\sqrt{(t-3)^2+4}}$$

let $x = t-3$
$dx = dt$

$$= \int \frac{dx}{\sqrt{x^2+4}}$$

let $x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$

Solve for these w/ a Δ.

$$= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

6.3. Partial Fractions

Ex. $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx$

$x^2+x-2 = (x+2)(x-1)$

So, $\dfrac{x+5}{x^2+x-2} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

or $x+5 = A(x-1) + B(x+2)$

$x=1: \quad 6 = 3B, \quad B = 2$

$x=-2: \quad 3 = -3A, \quad A = -1$

and $\dfrac{x+5}{x^2+x-2} = \dfrac{-1}{x+2} + \dfrac{2}{x-1}$

Then $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{-1}{x+2}\,dx + \int \frac{2}{x-1}\,dx = -\ln|x+2| + 2\ln|x-1| + C$

$= \boxed{\ln\left|\dfrac{(x-1)^2}{x+2}\right| + C}$.

Ex. $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r|l} & x^2+2 \quad + \dfrac{2}{x-1} \\ x-1 & x^3+0x^2+x+0 \\ & \underline{x^3 \qquad\;\; -x} \\ & \qquad 2x \\ & \qquad \underline{2x-2} \\ & \qquad\quad 2 \end{array}$$

$= \displaystyle\int x^2+2+\frac{2}{x-1}\,dx$

$= \boxed{\tfrac{1}{3}x^3 + 2x + 2\ln|x-1| + C}$

Ex. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x^2+4x-x-2)$$
$$= x[2x(x+2)-(x+2)]$$
$$= x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$

$x=0$: $\quad -1 = A(-1)(2) \qquad A=\frac{1}{2}$

$x=\frac{1}{2}$: $\quad \frac{1}{4}+1-1 = B(\frac{1}{2})(\frac{5}{2})$

$\qquad \frac{1}{4} = \frac{5B}{4} \qquad B=\frac{1}{5}$

$x=-2$: $\quad 4-4-1 = C(-2)(-5)$

$\qquad -1 = 10C \qquad C=\frac{-1}{10}$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{5}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

Ex. $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$ * Irreducible quadratic factors.

$$x^3 + 4x = x(x^2+4)$$

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

x^2 : $2 = A + B$ $B = 1$

x : $-1 = C$

#: $4 = 4A$ $A = 1$

so $$\int \frac{2x^2 - x + 4}{x^3+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+4}\,dx$$ 9/5/12

$$= \int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

Ex. $\displaystyle\int \frac{x^4+1}{x(x^2+1)^2}\,dx$ * repeated factors

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^4 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= A(x^4 + 2x^2 + 1) + x(Bx^3 + Cx^2 + Bx + C) + Dx^2 + Ex$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

$$= x^4(A+B) + x^3(C) + x^2(2A + B + D) + x(C+E) + (A)$$

$$A+B=1 \qquad A=1$$

$$c=0 \qquad C=E=B=0$$

$$D=-2$$

$$2A+B+D=0$$

$$C+E=0$$

$$A=1$$

$$\frac{1}{x} - \frac{2x}{(x^2+1)^2} = \frac{x^4+2x^2+1 \;-2x^2}{x(x^2+1)^2} = \frac{x^4+1}{x(x^2+1)^2}$$

So, $$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx + \int \frac{-2x}{(x^2+1)^2}\,dx = \boxed{\ln|x| + \frac{1}{x^2+1} + C}$$

$u = x^2+1$
$du = 2x\,dx$

<u>6.4</u> : Project only. Using WolframAlpha, calculators, and charts (in back of book) to help integrate. For example:

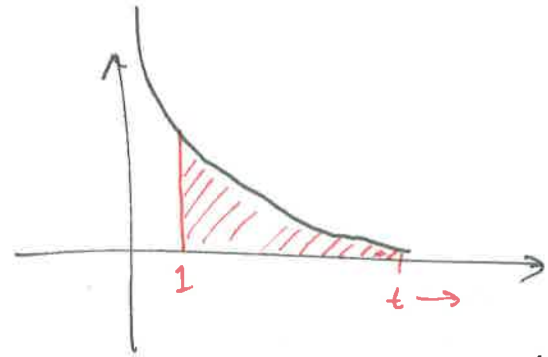
17. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

So, $$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+3^2} = \boxed{\frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C}$$

6.6 Improper Integrals

Case I. Integrals involving ∞.

Ex. $y = \frac{1}{x^2}$



For any $t > 1$ we can calculate $\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = \frac{1}{1} - \frac{1}{t}$

Reasonable conjecture: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

so $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - 0 = \boxed{1}$

Def'n. If $\int_a^t f(x)\,dx$ exist for all $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{provided the limit exists.}$$

If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx \quad \text{iff it exists.}$$

If these limits exist, then the integrals are said to be <u>convergent</u>, if not they are called <u>divergent</u>.

<u>Ex</u>. $\int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\ln x \Big|_1^t \right) = \lim_{t\to\infty} \left(\ln t - \ln 1 \right)$$

$$= \lim_{t\to\infty} \ln t - \lim_{t\to\infty} \ln 1$$

$$= \infty - 0 = \infty$$

so the integral <u>diverges</u>.

Thus $\int_1^\infty \frac{1}{x^2}\,dx$ converges $(=1)$, but $\int_1^\infty \frac{1}{x}\,dx$ diverges.

- Draw a picture.

$\boxed{\text{Exc}}$. $\int_1^\infty \frac{1}{x^n}\,dx$ Find out for what n this integral will converge.
(Exampl 4 in book)

<u>Ex</u>. $\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx = \lim_{t\to-\infty} \left(xe^x - e^x \Big|_t^0 \right)$

$$= \lim_{t\to-\infty} \left(0 - te^t - 1 + e^t \right)$$

$$= -\lim_{t\to-\infty} te^t - 1 + \underbrace{\lim_{t\to-\infty} e^t}_{0}$$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} \overset{L'H}{=} \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \frac{1}{-\infty} = 0$$

So, $\int_{-\infty}^0 xe^x\,dx = \boxed{-1}$ converges.

Ex. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

what to do w/ this?

$\boxed{\text{Exc.}}\boxed{\text{RE \#51}}$ Show that If $f(x)$ is odd, then $\int_{-t}^{t} f(x)\,dx = 0$ for all t. There is no reason that every odd function should converge. e.g. $f(x)=x$.

Def'n. If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)dx$$

i.e. $\int_{-\infty}^{\infty} f(x)dx$ converges iff it converges to both $\pm\infty$.

Ex. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}dx + \lim_{s\to\infty}\int_0^s \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\left(\arctan x \Big|_t^0\right) + \lim_{s\to\infty}\left(\arctan x \Big|_0^s\right)$$

$$= \arctan 0 - \lim_{t\to-\infty} \arctan t + \lim_{s\to\infty} \arctan s - \arctan 0$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) - 0 = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$ Converges. // 9/7/12

Project 2: §6.4

4.(a). $\int xe^x dx = e^x(x-1) + C$

Using Wolfram

$\int x^2 e^x dx = e^x(x^2-2x+2) + C$

$\int x^3 e^x dx = e^x(x^3-3x^2+6x-6) + C$

...

(c) Conjecture:

$$\int x^n e^x dx = e^x\left(x^n - \frac{d}{dx}(x^n) + \frac{d^2}{dx^2}(x^n) - + \cdots \frac{d^n}{dx^n}(x^n)\right) + C$$

where $\frac{d^n}{dx^n}(x^n) = n(n-1)(n-2)\cdots 1 = n!$

Math. induction:

1. initial case: $n=1$: $\int xe^x dx = e^x(x-1) + C = e^x\left(x - \frac{d}{dx}(x)\right) + C$ ✓
2. induction hypothesis: assume conjecture holds for some $n \geq 1$.
3. Prove for $n+1$:

$$\int x^{n+1} e^x dx = x^{n+1}e^x - \int \frac{d}{dx}\left[x^{n+1}\right] e^x dx$$ ← by Int. by parts

$$= x^{n+1}e^x - \int (n+1)x^n e^x dx$$

$$= x^{n+1}e^x - (n+1)\int x^n e^x dx$$ ← by Conj.

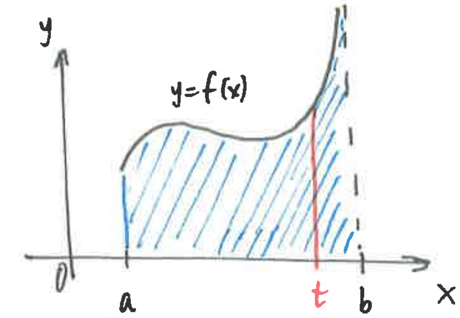
$$= x^{n+1}e^x - (n+1)e^x\left(x^n - \frac{d}{dx}(x^n) + \frac{d^2}{dx^2}(x^n) - + \cdots n!\right) + C$$

$$= x^{n+1}e^x - e^x\left[(n+1)x^n - (n+1)\frac{d}{dx}(x^n) + (n+1)\frac{d^2}{dx^2}(x^n) - + \cdots (n+1)n!\right] + C$$

$$= e^x\left[x^{n+1} - \frac{d}{dx}\left[x^{n+1}\right] + \frac{d^2}{dx^2}\left[x^{n+1}\right] - \frac{d^3}{dx^3}\left[x^{n+1}\right] + - \cdots (n+1)!\right] + C$$

$\square$

Case II. Discontinuous Integrands



If $\int_a^t f(x)\,dx$ exists for all $t \in (a,b)$, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx}$$

also left limits (from the right)

Ex. $\displaystyle \int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$ $\quad u = x-2$, $du = dx$ $\quad u(t) = t-2$, $u(5) = 3$

$$= \lim_{t\to 2^+} \int_{t-2}^{3} u^{-1/2}\,du$$

let $s = t-2$ so $s \to 0^+$ as $t \to 2^+$

$$= \lim_{s\to 0^+} \left[2u^{1/2} \Big|_s^3 \right]$$

$$= \lim_{s\to 0^+} \left[2\sqrt{3} - 2\sqrt{s} \right]$$

$$= 2\sqrt{3} - 0 = \boxed{2\sqrt{3}}$$

Ex. $\displaystyle \int_0^{\pi/2} \sec x\,dx$ $\qquad \sec\frac{\pi}{2}$ is undefined.

$$= \lim_{t\to \frac{\pi}{2}^-} \int_0^t \sec x\,dx = \lim_{t\to\frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| \Big|_0^t \right]$$

$$= \lim_{t\to\frac{\pi}{2}^-} \left[\ln|\sec t + \tan t| - \underbrace{\ln|\sec 0 + \tan 0|}_{=\ln 1 = 0} \right]$$

$$= \ln\left| \lim_{t\to\frac{\pi}{2}^-} \sec t + \lim_{t\to\frac{\pi}{2}^-} \tan t \right| = \infty \Rightarrow \underline{\text{diverges}}$$

Ex. $\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$

$$= \lim_{t\to 1^-}\left[\ln|x-1|\Big|_0^t\right] + \lim_{t\to 1^+}\left[\ln|x-1|\Big|_t^3\right]$$

$$= \underbrace{0 - \ln 0}_{+\infty} + \underbrace{\ln 2 - \ln 0}_{+\infty}$$

each piece diverges, so the whole diverges. // 9/10/12

The Comparison Theorem.

Suppose the f and g are continuous functions with

$f(x) \geq g(x) \geq 0$ for $x \geq a$

a) If $\int_a^\infty f(x)\,dx$ is convergent, then so is $\int_a^\infty g(x)\,dx$

b) If $\int_a^\infty g(x)\,dx$ is divergent, then so is $\int_a^\infty f(x)\,dx$.

Ex. $\int_0^\infty e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{Convergent}} + \int_1^\infty e^{-x^2}\,dx$

$e^{-x^2} \leq e^{-x}$ for all $x \geq 1$.

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t = \lim_{t\to\infty} \frac{-1}{e^t} + \frac{1}{e} = \boxed{\frac{1}{e}}$$

By comparison thm: $\int_1^\infty e^{-x^2}\,dx \leq \frac{1}{e}$ converges.

Using Complex analysis, one can show:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Ex. $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Ex. $\int_1^\infty \frac{dx}{x+e^{2x}}$. $\qquad x+e^{2x} > e^{2x}$ if $x \geq 1$

so $\frac{1}{x+e^{2x}} < \frac{1}{e^{2x}}$

and $\int_1^\infty \frac{dx}{x+e^{2x}} < \int_1^\infty \frac{1}{e^{2x}}\,dx$ which is convergent.

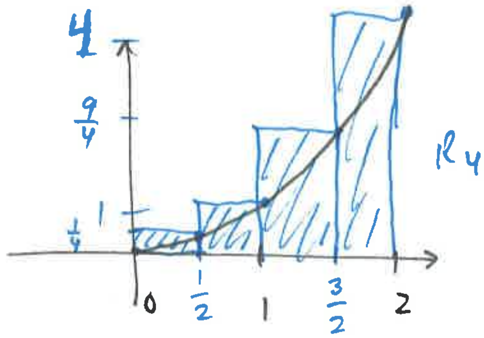
Back to §6.5: Approximate integration

6.5 Approximate Integration

Recall: Riemann Sum definition of an integral.

- Calc 1 integration notes added to web page. Worth looking at if you need a Riem. sum refresher. Figures missing. Draw them yourself or ask me what they are supposed to look like.

Ex. $\int_0^2 x^2\,dx$



$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$

$\Delta x_j = \begin{cases} \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} \\ x_i = a + i\Delta x \end{cases}$

$$R_4 = \sum_{j=1}^{4} f(x_j)\,\Delta x_j$$

$$= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{4}\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2} + \tfrac{9}{4}\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{8} + \tfrac{1}{2} + \tfrac{9}{8} + 2 = \frac{1+4+9+16}{8} = \boxed{\frac{30}{8}}$$ over estimate

$L_4 =$
$M_4 =$ } review.

Trapezoidal Rule:

$$\boxed{\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right] \quad \text{where } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x}$$

Trap for: $\int_0^2 x^2\,dx$

Error: Neither midpoint nor trap rule give the exact answer. The exact error is given by:

$$E_{M_n} = \int_a^b f(x)\,dx - M_n\,, \quad E_{T_n} = \int_a^b f(x)\,dx - T_n$$

Error bounds: Suppose $f''(x) \le K$ for all $a \le x \le b$. Then

$$|E_{T_n}| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_{M_n}| \le \frac{K(b-a)^3}{24n^2}.$$

In particular, this tells us how many n we need to use to <u>guarantee</u> a certain error level.

Ex We want to use T_n to estimate $\int_0^2 x^2\,dx$ with error at most $\frac{1}{100}$.

$$f''(x) = 2 = K, \quad \text{so} \quad |E_{T_n}| \le \frac{2(b-a)^2}{12n^2} \overset{\le}{\Rightarrow} \frac{1}{100}$$

$$\frac{1}{100} \ge \frac{2(2^3)}{12n^2} = \frac{16}{12n^2}$$

$$\Rightarrow n^2 \ge \frac{1600}{12} = \frac{400}{3} =$$

$$\text{so } n \ge \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}} \approx 11.5$$

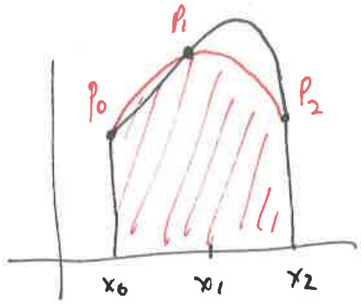
So we must use <u>at least</u> 12 boxes to get this accuracy. Since we're lazy, we should use exactly 12.

On computer: $\int_0^1 e^{-x^2}\,dx$

#5 $\int_0^{\pi} x^2 \sin x\,dx$ $n=8$ M, T, S

9/12/12

Simpson's Rule: A combination of T_n and M_n (Ex. 40).



Uses a parabola that passes thru P_0, P_1, and P_2.

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

where n is even and $\Delta x = \frac{b-a}{n}$

Exs above.

Proof / Derivation in book pp. 338-9.

Error bound for S_n:

Suppose $|f^{(4)}(x)| \le K$ for $a \le x \le b$. Then

$$|E_{S_n}| \le \frac{K(b-a)^5}{180\,n^4}$$

Ex (20). How large must n be to guarantee that Simpsons rule approximates $\int_0^1 e^{x^2}\,dx$ to within 0.00001?