§10.5: Equations of lines and planes in space

Recall that any point in space determines a vector

$$P(x,y,z) \longmapsto \langle x,y,z\rangle = \vec{r}$$

we call this vector the <u>position vector</u> for P.

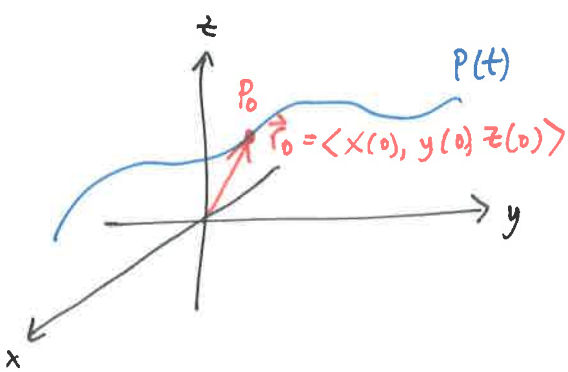
If we think of P as a particle that is moving, then the path of p is a parametrized curve in space

$$P(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$$

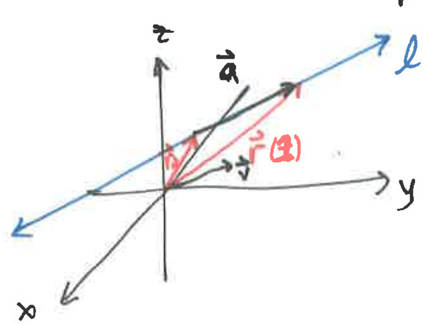
Then we can write the position vector as a parametrized (time-dependent) vector:

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

where $\vec{r}(0) = \vec{r}_0$ is the <u>initial position</u> of the particle.



<u>Special Case</u> A line ℓ in space



$\vec{v}$ is the direction vector of the line.
if It's a unit vector:
$\vec{v} = \frac{\vec{a}}{\|\vec{a}\|}$.

Then the line can be described by

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}} \qquad \text{vector eq'n.}$$

But $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

If $\vec{v} = \langle a, b, c \rangle$, then we can write out 3 parametric equations for ℓ:

$$\begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases} \qquad \text{parametric eq'ns.}$$

<u>Ex.</u> Find both for the line passing thru $(5,1,3)$ parallel to $\langle 1, 4, -2 \rangle$

$$\vec{r}(t) = \langle 5, 1, 3 \rangle + \langle t, 4t, -2t \rangle$$

$$= \langle 5+t, 1+4t, 3-2t \rangle$$

So, Parametrically: $\begin{cases} x(t) = 5+t \\ y(t) = 1+4t \\ z(t) = 3-2t \end{cases}$

For any value of t we get a point on ℓ.

$$\vec{r}(-1) = \langle 5-1, 1-4, 3+2 \rangle = \langle 4, -3, 5 \rangle$$

so $(4,-3,5)$ is on the line.

Thus, we could write ℓ as

$$\vec{r} = \langle 4+t, -3+4t, 5-2t \rangle$$

so ℓ can have <u>many</u> parametric representations.

To obtain symmetric equations, solve for t and set 'em all equal:

$$x = x_0 + at \Rightarrow t = \frac{x - x_0}{a}$$

$$y = y_0 + bt \Rightarrow t = \frac{y - y_0}{b}$$

$$z = z_0 + ct \Rightarrow t = \frac{z - z_0}{c}$$

Symm. Eqn's are:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Ex. Find all three for the line that passes through $A(2,4,-3)$ and $B(3,-1,1)$.

$$\vec{v} = \langle 1, -5, 4 \rangle$$

vec. $\boxed{\vec{r}(t) = \langle 2+t,\ 4-5t,\ -3+4t \rangle}$

par. ~~Symm.~~ $\boxed{\begin{cases} x = 2 + t \\ y = 4 - 5t \\ z = -3 + 4t \end{cases}}$

Symm. $\boxed{x - 2 = \frac{y - 4}{-5} = \frac{z + 3}{4}}$

Q. At what point does this line intersect the x-y plane?

$z = 0$ when $t = \frac{3}{4}$

$x(3/4) = 2 + 3/4 = 11/4$

$y(3/4) = 4 - 5(3/4) = 1/4$

so intersection is $P(11/4, 1/4, 0)$

Line segments:

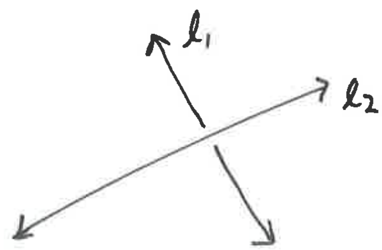
$A(x_0, y_0, z_0) = \vec{r}_0$ $\qquad$ $B(x_1, y_1, z_1) = \vec{r}_1$

The line segment is given in vector form by

$$\vec{r}(t) = (1-t)\,\vec{r}_0 + t\,\vec{r}_1 \,, \quad 0 \le t \le 1$$

so, $\vec{r}(0) = \vec{r}_0$, $\vec{r}(1) = \vec{r}_1$

In 3D it is possible to have two lines that do not intersect and are not parallel. These are called <u>skew</u> <u>lines</u>



<u>Recall</u>. Two lines in 3D are parallel if their direction vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel (multiples).

<u>Ex</u>. show that l_1 and l_2 are skew:

l_1: $\vec{r}_1(t) = \langle 1+t, -2+3t, 4-t \rangle$ $\qquad$ $\vec{v}_1 = \langle 1, 3, -1 \rangle$

l_2: $\vec{r}_2(t) = \langle 2s, 3+s, -3+4s \rangle$ $\qquad$ $\vec{v}_2 = \langle 2, 1, 4 \rangle$

$\vec{v}_1 \not\parallel \vec{v}_2$. $\qquad$ $s = \frac{1+t}{2} \Rightarrow -2+3t = 3 + \frac{1+t}{2}$

$-4+6t = 6+1+t$

$-11 = -5t$

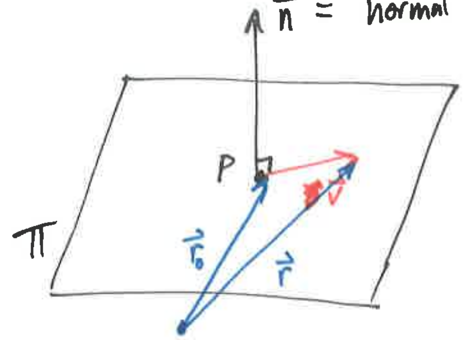
$t = \frac{11}{5}$

$\Rightarrow s = \frac{8}{5}$

<u>But</u>:

$4 - \frac{11}{5} \ne -3 + \frac{32}{5}$

<u>Planes</u> are more complicated.



$P = \vec{r}_0 = (x_0, y_0, z_0)$
$\vec{v}$ is in the plane Π

The vector $\vec{v} = \vec{r} - \vec{r}_0$ is $\perp$ to $\vec{n}$, so

$$\vec{n} \cdot \vec{v} = 0. \overset{\text{or}}{\Rightarrow} \boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$$

All of the points $\vec{r} = \langle x, y, z \rangle$ that satisfy this eq'n determine the plane.

$$\vec{n} = \langle a, b, c \rangle \neq \langle 0, 0, 0 \rangle$$
$$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

so eq'n of plane is:

$$\vec{n} \cdot \vec{v} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

OR

$$\boxed{ax + by + cz + d = 0}$$

w/ $d = -(ax_0 + by_0 + cz_0)$

Ex. Find eq'n of plane passing thru $P(1,3,2), Q(3,-1,6), R(5,2,0)$.

$\vec{r}_0 = \langle 1, 3, 2 \rangle$

$\vec{v} = \vec{PQ} = \langle 2, -4, 4 \rangle \rightleftharpoons \langle 1, -2, 2 \rangle$

~~so $\vec{r} = \langle 1+t, 3-2t, 2+2t \rangle$~~

$\vec{u} = \vec{QR} = \langle 2, 3, -6 \rangle$

$$\vec{n} = \vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 2 & 3 & -6 \end{vmatrix} = \vec{i}(6) - \vec{j}(-10) + \vec{k}(7)$$

$$= \langle 6, 10, 7 \rangle$$

Plane is thus

$$6(x-1) + 10(y-3) + 7(z-2) = 0$$

$$6x + 10y + 7z = 6 + 30 + 14$$

$$\boxed{6x + 10y + 7z = 50}$$

Def'n. Two planes are $\parallel$ if their normal vectors are $\parallel$.

If not $\parallel$, then the angle between the planes is the acute $\measuredangle$ between normal vectors.

Two planes intersect in a line.

Ex. Find the $\measuredangle$ between $x+y+z=1$ and $x-2y+3z=1$

$\vec{n}_1 = \langle 1,1,1 \rangle \qquad \vec{n}_2 = \langle 1,-2,3 \rangle$

$$\measuredangle = \theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}\right) = \arccos\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

$\|\vec{n}_1\| = \sqrt{3}$ $\qquad \vec{n}_1 \cdot \vec{n}_2 = 1 - 2 + 3 = 2$

$\|\vec{n}_2\| = \sqrt{14}$

Find symmetric eqns for ℓ, the line where Π_1 and Π_2 intersect.

the line ℓ is $\perp$ to both $\vec{n}_1$ and $\vec{n}_2$, by defn.

So it has dir. vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \vec{i}(5) - \vec{j}(2) + \vec{k}(-3)$$

$$= \langle 5, -2, -3 \rangle$$

We also need a point in the intersection.

set $z=0$: $\begin{cases} x+y=1 \\ x-2y=1 \end{cases} \Rightarrow$ $x = 1-y$

$1-y-2y=1$

$3y=0$

$y=0$

$\Rightarrow x=1$

So $(1,0,0)$ is a pt in the line ℓ.

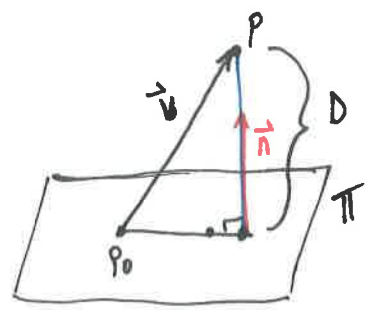
Then ℓ is given by

$$\vec{r}(t) = \langle 1+5t, -2t, -3t \rangle$$

or

$$\boxed{\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}}$$

Distance between a pt. and a plane:



$D =$ length of proj of $\vec{v}$ onto $\vec{n}$

$= \| \text{proj}_{\vec{n}} \vec{v} \| = | \text{comp}_{\vec{n}} \vec{v} |$

recall: $\text{proj}_{\vec{n}} \vec{v} = \dfrac{\vec{n} \cdot \vec{v}}{\vec{n} \cdot \vec{n}} \vec{n}$ and

$$\text{comp}_{\vec{n}} \vec{v} = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|}$$

here $\vec{v} = \vec{PP_0} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$\vec{n} = \langle a, b, c \rangle$

$$\text{comp}_{\vec{n}} \vec{v} = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}}$$

so $\boxed{D = \dfrac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}}$

Ex. Find D between $\left.\begin{array}{l} 10x + 2y - 2z = 5 \\ \text{and } 5x + y - z = 1 \end{array}\right\}$ parallel planes

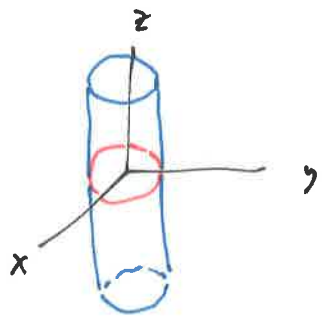
Find any pt in Π_1 and find its D to Π_2.

§10.6. Quadratic Surfaces and Cylinders

To graph surfaces, we look at cross-sections called <u>traces</u>. The cross-sections are parallel to coordinate planes.

<u>Ex</u>. $x^2 + y^2 = 1$

in xy-plane : circle z can be anything

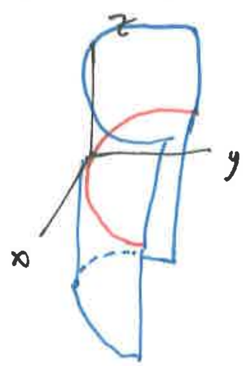


So we get an infinite right circular cylinder.

<u>Ex</u>. $y^2 + z^2 = 1$

<u>Ex</u>. $y = x^2$

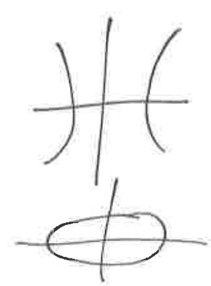
In xy-plane: parabola z: anything



This is also a ~~cyclo~~ "cylinder", but it's a parabolic one.

other choices, like $x^2 - y^2 = 1$

$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$



Quadratic Surfaces

Equations like ~~$\frac{(x-h)^2}{a^2}$~~

~~$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + D$~~

all combos of 2nd order x, y, z's.

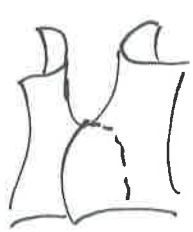
Ex. $x^2 + y^2 + z^2 = r^2$

sphere of radius r

Ex. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ ellipsoid

Ex. $z = 4x^2 + y^2$ elliptic-paraboloid

Ex. $z = y^2 - x^2$ hyperbolic paraboloid or saddle surface



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