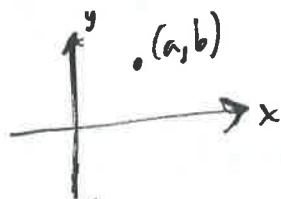


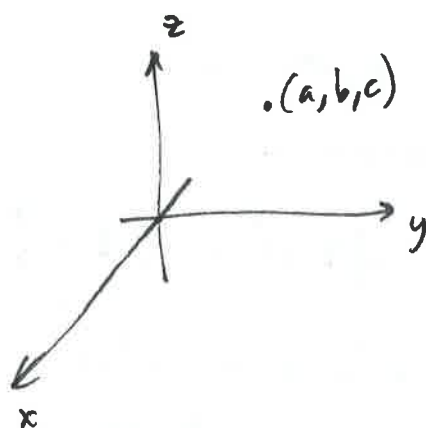
10.1: 3D coordinates

In the 2D Cartesian plane any point can be represented by an ordered pair (a,b) , where a is the x-coord and b is the y-coord.



In 3D, we need an ordered triple: (a,b,c)

Standard coord axes:

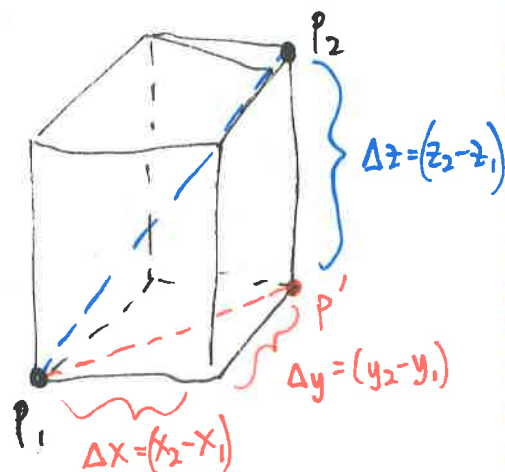
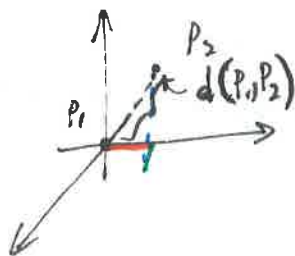


Use the corner of the room to help visualize.

The axes divide the space into "octants". The first octant has $x,y,z > 0$.

Examples: describe: $y=5$, $y=x$

3D Distance formula:



$$P_1 = (x_1, y_1, z_1)$$

$$P' = (x_2, y_2, z_1)$$

$$P_2 = (x_2, y_2, z_2)$$

Using Pythagorean Thm:

$$d(P_1, P_2) = \sqrt{d(P_1, P')^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example. Let $P=(x,y,z)$ be a "variable point" and $P_0=(x_0,y_0,z_0)$ be a fixed point. Describe this set:

$$d(P, P_0) = r$$

Eqn of a sphere: An eqn of a sphere centered at $C=(h,k,l)$ with radius r is:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Ex. make the center = origin.

Ex. What region is represented by $1 \leq x^2 + y^2 + z^2 \leq 4$?
what is its volume?

$$\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3 = \frac{32}{3}\pi - \frac{4}{3}\pi = \boxed{\frac{28}{3}\pi \text{ units}^3}$$

Ex. (14.) Find the center and radius of the sphere:

$$x^2 + y^2 + z^2 = 4x - 2y$$

$$x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 = 4 + 1$$

$$(x-2)^2 + (y+1)^2 + z^2 = 5$$

$$\boxed{C = (2, -1, 0) \quad r = \sqrt{5}}$$

Ex. ⁽²⁰⁾ Find eqn of largest sphere w/ center $C(5, 4, 9)$ contained in the first octant:

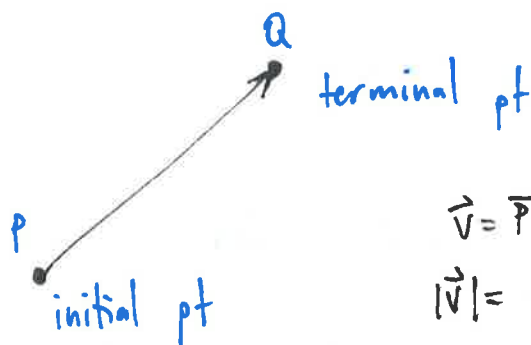
$$\boxed{(x-5)^2 + (y-4)^2 + (z-9)^2 = 16}$$

10.2: Vectors

In engineering and many sciences:

A vector is a quantity that has both magnitude and direction.

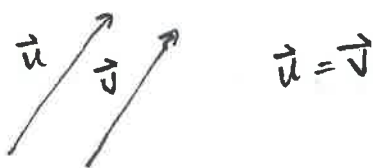
Ex. displacement, velocity, acceleration, force.



$$\vec{v} = \overrightarrow{PQ}$$

$$|\vec{v}| = d(P, Q) = \text{magnitude (or length) of } \vec{v}.$$

Two vectors w/ the same length and direction are "the same".
The initial and terminal pts don't really matter.

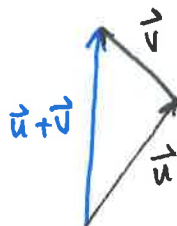


Because of this, we have a property called parallel translation, or "pick-and-plop." We can move vectors around as we please, as long as we don't change the length or direction.

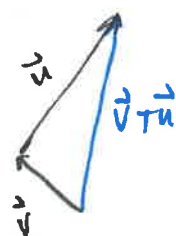
Vector Addition: "tip-to-tail" * [Triangle & Parallelogram Laws]



$$\vec{u} + \vec{v} =$$



$$\vec{v} + \vec{u}$$



$$\vec{u} + \vec{v} = \vec{v} + \vec{u} : \text{Commutative!}$$

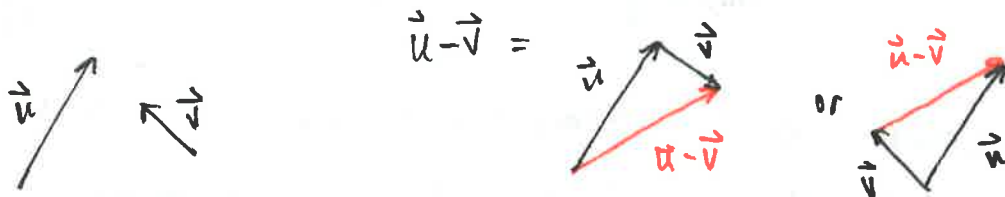
Scalar Multiplication: Let $c > 0$, and $\vec{v}$ a vector. Then $c\vec{v}$ is a vector in the same direction as $\vec{v}$, but with length $c|\vec{v}|$.

If $c = -1$, then $c\vec{v}$ switches the tip and tail.

So for $c < 0$, $c\vec{v}$ is in the "opposite" direction, w/ length $|c||\vec{v}|$.

If $c = 0$ or $\vec{v} = \vec{0}$, then $c \cdot \vec{v} = \vec{0}$.

Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ Not commutative!

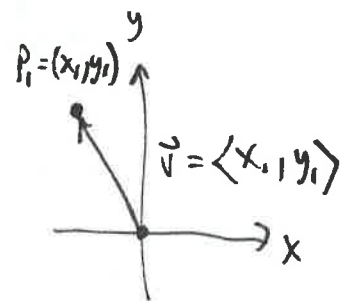
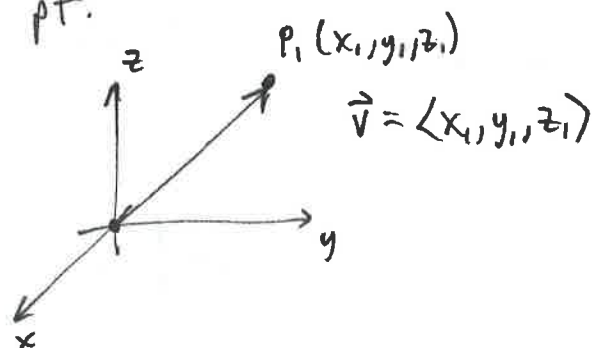


Ex. $\vec{u} - 2\vec{v}$, $\vec{v} - \vec{u}$.

(*) Notice, this was all done w/out coordinates!

Components and Standard Position:

We can put the initial pt of every vector at the origin (bc of parallel translation), then a vector is just described by its terminal pt.



We call the space of all 3-vectors V^3 , and all 2-vectors V^2 .
n-vectors: V^n .

Given initial and terminal pts only:

$$A = (a_1, a_2, a_3)$$

$$B = (b_1, b_2, b_3)$$

$$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

} Component form is terminal minus initial.

We can write $\vec{v} = \vec{AB}$, if we want, ~~also~~ $\vec{v} \in V^3$.

Ex. $A = (2, -3, 4)$, $B = (-2, 1, 1)$, Find $\vec{AB} = \vec{v}$.

Defn. If $\vec{v} \in V^3$ is in standard position; i.e., $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in comp. form,

then $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the length of $\vec{v}$.

For $\vec{v} \in V^n$, ~~also~~ $|\vec{v}| = \sqrt{v_1^2 + \dots + v_n^2}$. Also, $\vec{v} \in V^2$.

Addition, Subtraction, and Scalar Mult. in comp. form:

We'll write the formulas in V^2 .

$$\vec{a} = \langle a_1, a_2 \rangle \quad \vec{b} = \langle b_1, b_2 \rangle$$

$$\begin{cases} \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \\ \vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle \\ c\vec{a} = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle \end{cases}$$

Exc. Write the formulae for V^n .

Ex. $\vec{a} = \langle 4, 0, 3 \rangle$ Find: $|\vec{a}|$, $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, $2\vec{a} + 5\vec{b}$.
 $\vec{b} = \langle -2, 1, 5 \rangle$

Properties of Vectors: Let $\vec{a}, \vec{b}, \vec{c} \in V^n$, $c, d \in \mathbb{R}$, then

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ addition is commutative

2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ add. is ass.

3. $\vec{a} + \vec{0} = \vec{a}$ $\vec{0}$ is add. identity

4. $\vec{a} + (-\vec{a}) = \vec{0}$ vectors have add. inverses

5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$ scal. mult. dist. over add.

6. $(c+d)\vec{a} = c\vec{a} + d\vec{a}$ scalar mult. dist over scalar add.

7. $(cd)\vec{a} = c(d\vec{a})$ scal. mult. is ass.

8. $1 \cdot \vec{a} = \vec{a}$ $1 \in \mathbb{R}$ is scal. mult. id.

* Exc. Try to prove (or at least verify) these using component form. Try it in V^2 or V^3 first if V^n is too confusing.

→ Verification^{of (2)} using Parallelogram Law in book (p. 527). we already did it.
wed. 11

Yet another way:

Standard basis vectors:

In V^3 , let $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$.

We can write any other vector in V^3 as a linear combination of these vectors.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

Then $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

Ex. $\vec{a} = \langle 5, -3, 2 \rangle$

$$\vec{a} = 5\vec{i} - 3\vec{j} + 2\vec{k}$$

Ex. (16) $\vec{a} = 2\vec{i} - 4\vec{j} + 4\vec{k}$

$\vec{b} = 2\vec{j} - \vec{k}$

Find: $\vec{a} + \vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$, $|\vec{a} - \vec{b}|$.

Ex. ⁽¹⁸⁾ A unit vector is a vector of length 1.

- If $\vec{v} = \langle -2, 4, 2 \rangle$, find a unit vector in the direction of $\vec{v}$.
- Find a vector of length 6 in the direction of $\vec{v}$.

Ex. ^{-ish} (19) If $\vec{u} \in \mathbb{R}^2$ is in quadrant III and makes a $\pi/4$ angle w/ the negative x-axis, find its comp. form.

Note: Any vector in $\mathbb{R}^2$ can be written as

$$\vec{v} = r \langle \cos \theta, \sin \theta \rangle \quad \text{where } r = |\vec{v}| \text{ and } \theta \text{ is } \angle \text{ from standard pos.}$$

10.3: The Dot Product

Def'n. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. i.e., $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \leftarrow \text{A number!}$$

* Also called scalar product or inner product

Defn. In V^n , $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$.

Ex. $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 6 - 4 = 2$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = -6 + 14 - 2 = 6$$

$$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k}) = \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle = 0 + 4 + 3 = 7.$$

Th. Properties of the dot product. If $\vec{a}, \vec{b}, \vec{c} \in V^n$ and $k \in \mathbb{R}$, then

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

• is comm

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

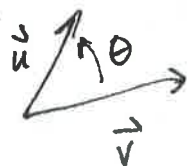
• distributes over +

4. $(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$

5. $\vec{0} \cdot \vec{a} = 0$

Prove some.

Geometrically, $\cdot$ is related to the angle between two vectors.



Th. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta.$$

See book for proof.

8/22/12

Ex. If $|\vec{a}| = 4$, $|\vec{b}| = 6$, $\theta = \pi/3$, find $\vec{a} \cdot \vec{b}$. 12.

$$\vec{a} \cdot \vec{b} = 4 \cdot 6 \cdot \cos(\pi/3) = 24 \cdot (1/2) = 12.$$

Ex. Find the angle between $\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta \\ \Rightarrow \cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \Rightarrow \theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) \end{aligned}$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \langle 2, 2, -1 \rangle \cdot \langle 5, -3, 2 \rangle = 10 - 6 - 2 = 2 \\ |\vec{a}| &= \sqrt{4+4+1} = 3 \\ |\vec{b}| &= \sqrt{25+9+4} = \sqrt{38} \end{aligned} \quad \left. \vphantom{\begin{aligned} \vec{a} \cdot \vec{b} \\ |\vec{a}| \\ |\vec{b}| \end{aligned}} \right\} \cos \theta = \frac{2}{3\sqrt{38}}$$

$$\text{so } \theta = \cos^{-1} \left(\frac{2}{3\sqrt{38}} \right) \approx 1.46$$

Ex. Find the angle between $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$
 $\vec{v} = 5\vec{i} - 4\vec{j} + 2\vec{k}$

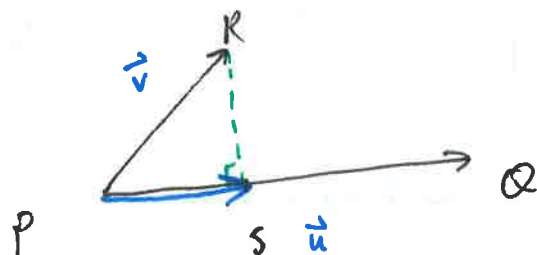
$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 10 - 8 - 2 = 0 \\ |\vec{u}| &= 3 \\ |\vec{v}| &= \sqrt{25+16+4} = \sqrt{45} \end{aligned} \quad \left. \vphantom{\begin{aligned} \vec{u} \cdot \vec{v} \\ |\vec{u}| \\ |\vec{v}| \end{aligned}} \right\} \begin{aligned} \cos \theta &= \frac{0}{3\sqrt{45}} \\ \cos \theta &= 0 \\ \theta &= \cos^{-1}(0) = \frac{\pi}{2} \\ \Rightarrow \vec{u} &\perp \vec{v}. \end{aligned}$$

Th. Two vectors $\vec{a}, \vec{b} \in V^n$ are orthogonal iff $\vec{a} \cdot \vec{b} = 0$.

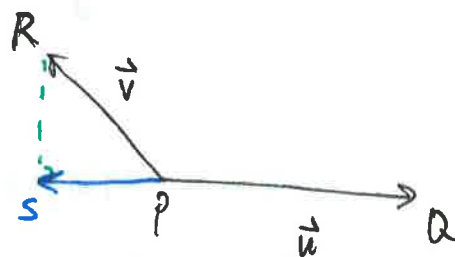
More: if $\vec{a} \cdot \vec{b} > 0$, then $0 \leq \theta < \frac{\pi}{2}$

if $\vec{a} \cdot \vec{b} < 0$, then ~~then~~ $\frac{\pi}{2} < \theta \leq \pi$

Projections:



$$\vec{PS} = \text{proj}_{\vec{u}} \vec{v}$$



$$\vec{PS} = \text{proj}_{\vec{u}} \vec{v}$$

Notice also: $\vec{PR} = \vec{PS} + \vec{SR}$

($\vec{PS}$ projection of $\vec{v}$ onto $\vec{u}$.)

where $\vec{SR} = \vec{v} - \text{proj}_{\vec{u}} \vec{v} = \text{orth}_{\vec{u}} \vec{v}$

and $\vec{PS} \perp \vec{SR}$

So we can decompose $\vec{PR}$ into orthogonal components w/ one in the direction of $\vec{PQ}$.

Defn. $\vec{u}, \vec{v} \in V^n$.

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}}_{\#} \vec{u}$$

Ex. $\vec{u} = \langle -2, 3, 1 \rangle$ $\vec{v} = \langle 1, 1, 2 \rangle$

$$\text{proj}_{\vec{u}} \vec{v} = ? = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$\vec{u} \cdot \vec{v} = \langle -2, 3, 1 \rangle \cdot \langle 1, 1, 2 \rangle = -2 + 3 + 2 = 3$$

$$\vec{u} \cdot \vec{u} = \langle -2, 3, 1 \rangle \cdot \langle -2, 3, 1 \rangle = 4 + 9 + 1 = 14$$

$$\text{proj}_{\vec{u}} \vec{v} = \frac{3}{14} \langle -2, 3, 1 \rangle = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

The scalar projection of $\vec{u}$ on $\vec{v}$ is the ~~length~~ coefficient length and direction $\text{proj}_{\vec{u}} \vec{v}$.

Defn. $\vec{u}, \vec{v} \in V^n$. $\text{comp}_{\vec{u}} \vec{v} = \left\| \text{proj}_{\vec{u}} \vec{v} \right\| = \left\| \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} \right\| = \left| \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \right| \left| \frac{\vec{u}}{|\vec{u}|} \right| = \left| \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \right|$

$$\text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|}$$

Ex. ⁽²⁴⁾ $\vec{a} = \langle 1, 2 \rangle$ $\vec{b} = \langle -4, 1 \rangle$ Find $\text{comp}_{\vec{a}} \vec{b}$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-4 + 2}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

Prove: $\text{orth}_{\vec{u}} \vec{v} \perp \text{proj}_{\vec{u}} \vec{v}$ and

Ex (43). Cauchy - Schwarz Inequality: $\vec{u}, \vec{v} \in V^n$

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$$

Ex (44). Triangle Inequality: $\vec{u}, \vec{v} \in V^n$

$$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$$

pf. $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$$\begin{aligned} &= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v} \\ &\leq (|\vec{u}| + |\vec{v}|)^2 \end{aligned}$$

$$\Rightarrow |\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}| \quad \square$$

Ex (45). Parallelogram Law: $\vec{u}, \vec{v} \in V^n$

$$|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$$

10.4: The Cross Product

Defn. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

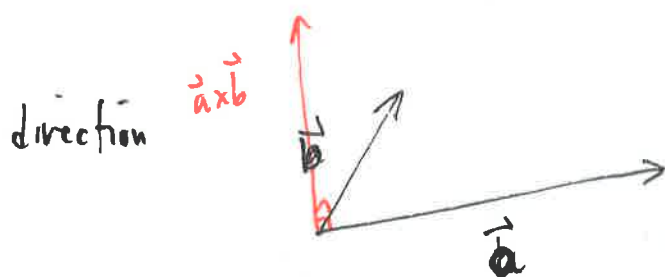
Or it can be done using determinants.

Ex. $\vec{a} = \langle 1, 3, 4 \rangle$ $\vec{b} = \langle 2, 7, -5 \rangle$

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \\ &= (-15 - 28)\vec{i} - (-5 - 8)\vec{j} + (7 - 6)\vec{k} \\ &= -43\vec{i} + 13\vec{j} + \vec{k} = \langle -43, 13, 1 \rangle \end{aligned}$$

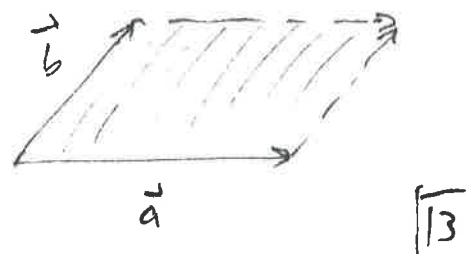
FTIS: $\vec{a} \times \vec{a} = \vec{0}$ for any $\vec{a} \in V^3$.

Geometrically:



Right hand rule

length: $|\vec{a} \times \vec{b}| = \text{area of parallelogram}$



$\sqrt{13}$

Ex. (26) Find a nonzero vector orthogonal to the plane containing

$$P(2, 5)$$

$$Q(-1, 3, 4)$$

$$R(3, 0, 6)$$

$$\vec{PQ} = \langle 3, -2, 1 \rangle$$

$$\vec{QR} = \langle -4, 3, -2 \rangle$$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -4 & 3 & -2 \end{vmatrix} = \langle 1, +2, 1 \rangle$$

Find the area of ΔPQR :

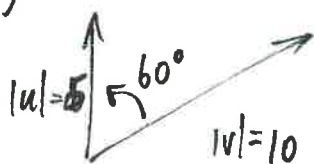
$$A = \frac{1}{2} |\vec{PQ} \times \vec{QR}| = \frac{1}{2} \sqrt{1+4+1} = \boxed{\frac{1}{2} \sqrt{6}}$$

9/24/12

Th. If θ is the angle between $\vec{a}$ and $\vec{b}$ ($0 \leq \theta \leq \pi$), then

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

Ex. (10)



Find $|\vec{u} \times \vec{v}|$

$$= 5 \cdot 10 \cdot \sin\left(\frac{\pi}{3}\right) = 50 \left(\frac{\sqrt{3}}{2}\right) = 25\sqrt{3}$$

$$60^\circ = \frac{\pi}{3} \text{ rad}$$

Cor. $\vec{a}, \vec{b} \in V^3 \neq 0 \quad \vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0} \quad \square$

✓ Properties: $\vec{a}, \vec{b}, \vec{c} \in V^3$ $c \in \mathbb{R}$

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ "anti ~~symmetric~~ ^{commutative}"

2. $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$

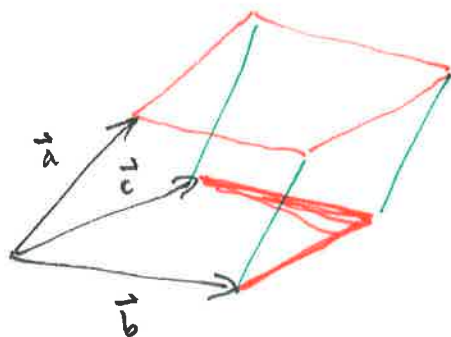
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ "Scalar Triple Product"

6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

The triple product has geometric meaning:



Parallelepiped

The volume of this parallelepiped is

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Ex. (30) $\vec{a} = \vec{i} + \vec{j} - \vec{k} = \langle 1, 1, -1 \rangle$ Find vol. of parallelepiped.

$$\vec{b} = \vec{i} - \vec{j} + \vec{k} = \langle 1, -1, 1 \rangle$$

$$\vec{c} = -\vec{i} + \vec{j} + \vec{k} = \langle -1, 1, 1 \rangle$$

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \begin{vmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(-2) - 1(2) - 1(0) \\ &= -2 - 2 \\ &= -4 \end{aligned}$$

so, volume = 4.