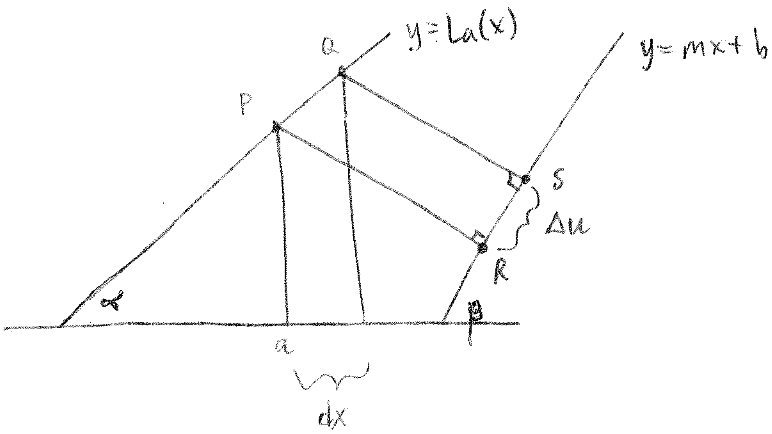
(1)



$P(a, f(a))$

$Q(a+dx,\ f(a)+dy)$

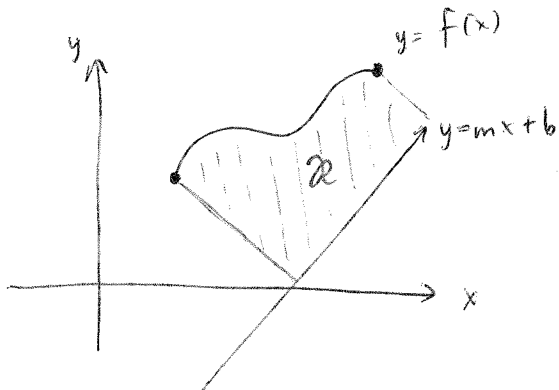
$dy = f'(a)\,dx$

$y = L_a(x)$ is the linearization of $y=f(x)$ at $x=a$; i.e., the equation of the tangent line:

$$L_a(x) = f'(a)(x-a) + f(a).$$

What are we trying to do?

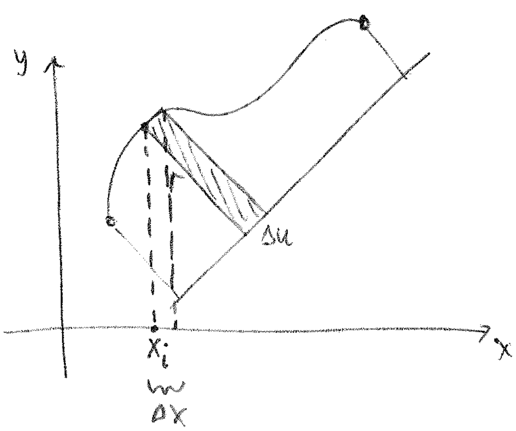
(2)



Find the area of the region $\mathcal{R}$

To do so:

(3)



Add up the area of rectangles, then take a limit (Riemann sum)

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} r \cdot \Delta u$$

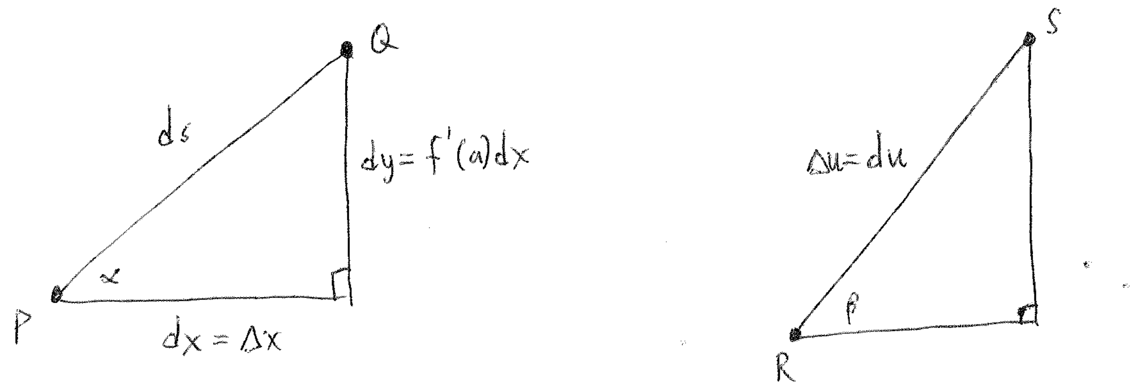
But we need $r = r(x)$ and Δu to be functions of x and Δx.

Instead of using the curve itself to solve for r and Δu, we use the linearization (tangent line) from figure (2).

From here we see that $r(x) = \text{dist}(P,R)$ and
$\Delta u = \text{dist}(R,S)$.

First we deal w/ Δu. Consider the right triangles:

(4)



By the Pythagorean Theorem $ds^2 = dx^2 + (f'(a))^2 dx^2$

$$\Rightarrow ds = \sqrt{1+(f'(a))^2}\, dx$$

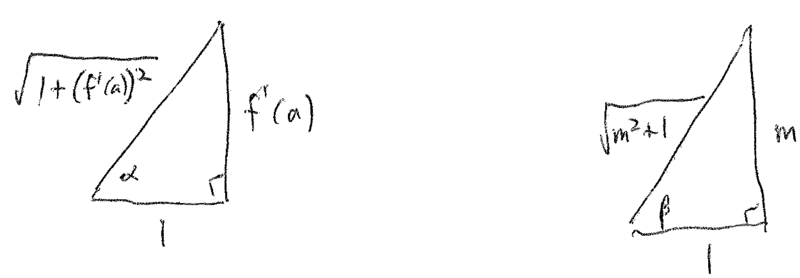
(This is the arc length element from section 7.4! :))

Because the hypotenuses are segments of the lines we have

$\tan \alpha = f'(a)$ (slopes of the lines)
$\tan \beta = m$

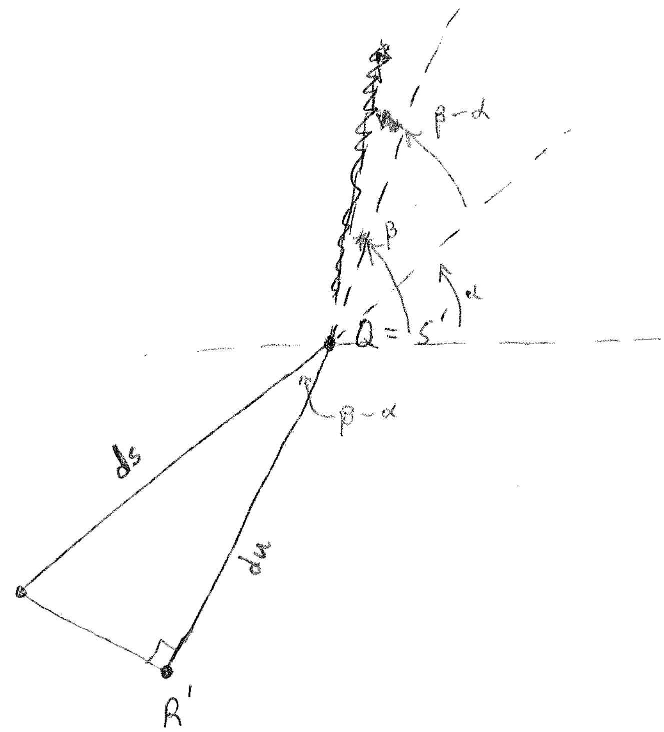
We can draw representative Δs for α, β:

(5)



Now make a new right triangle by collapsing $S \to Q = S'$ and $R \to R'$ in figure (1):

(6)



Then $\cos(\beta-\alpha) = \dfrac{du}{ds}$ or $du = \cos(\beta-\alpha)\, ds$.

By a sum/difference trig identity:

$$\cos(\beta-\alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha$$

Use the diagrams in (5) to obtain

$$\cos(\beta-\alpha) = \frac{1}{\sqrt{m^2+1}} \cdot \frac{1}{\sqrt{1+(f'(a))^2}} + \frac{m}{\sqrt{m^2+1}} \cdot \frac{f'(a)}{\sqrt{1+(f'(a))^2}}$$

$$= \frac{1 + mf'(a)}{\sqrt{m^2+1}\,\sqrt{1+(f'(a))^2}}$$

and $du = \cos(\beta-\alpha)\, ds$

$$= \frac{(1+mf'(a))}{\sqrt{m^2+1}\,\cancel{\sqrt{1+(f'(a))^2}}} \cancel{\sqrt{1+(f'(a))^2}}\, dx = \boxed{\frac{1}{\sqrt{m^2+1}}(1+mf'(a))\, dx = du} \quad (*)$$

so we're half way done :).

Now we need to find $r = \text{dist}(P,R)$.

We know $P = (a, f(a))$ from figure (1).

R is the intersection of two perpendicular lines.

One is $y = mx + b$

the other has slope $\frac{-1}{m}$ and passes thru P.

so it's given by $y = -\frac{1}{m}(x-a) + f(a)$

set these equal and solve for x & y:

$$-\frac{1}{m}x + \frac{1}{m}a + f(a) = mx + b$$

$$-x + a + mf(a) = m^2 x + mb$$

$$(m^2+1)x = a + mf(a) - mb$$

$$\left.\begin{aligned} x &= \frac{1}{m^2+1}\left[a + mf(a) - mb\right] \\ y = mx+b &= \frac{m}{m^2+1}\left[a + mf(a) - mb\right] + b \end{aligned}\right\} R = (x,y)$$

Now,

$$r = \text{dist}(P,R) = \sqrt{\left(\frac{1}{m^2+1}\left[a + mf(a) - mb\right] - a\right)^2 + \left(\frac{m}{m^2+1}\left[a + mf(a) - mb\right] + b - f(a)\right)^2}$$

Now, we "just" need to carefully simplify this. We'll ignore the square root for now, and first find r^2.

$$r^2 = \left(\frac{\cancel{a} + mf(a) - mb - am^2 - \cancel{a}}{m^2+1}\right)^2 + \left(\frac{ma + \cancel{m^2 f(a)} - \cancel{m^2 b} + \cancel{m^2 b} + b - \cancel{m^2 f(a)} - f(a)}{m^2+1}\right)^2$$

$$r^2 = \frac{m^2\left(f(a)-b-ma\right)^2 + \left(-f(a)+b+ma\right)^2}{(m^2+1)^2}$$

$$= \frac{m^2\left(f(a)-ma-b\right)^2 + 1\left(f(a)-ma-b\right)^2}{(m^2+1)^2}$$

$$= \frac{\cancel{(m^2+1)}\left(f(a)-ma-b\right)^2}{(m^2+1)^{\cancel{2}}}$$

So, $\boxed{r = \frac{1}{\sqrt{m^2+1}}\left[f(a)-ma-b\right]}$

Now, recall that $\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} r(x)\,\Delta u(x_i)$.

Letting $a = x_i$ in our formulas, we have

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{m^2+1}}\left[f(x_i)-mx_i-b\right]\cdot\frac{1}{\sqrt{m^2+1}}\cdot\left[1+mf'(x_i)\right]\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{m^2+1}\left(f(x_i)-mx_i-b\right)\left(1+mf'(x_i)\right)\Delta x$$

But this is a Riemann Sum! So it becomes

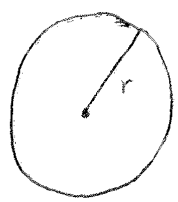
$$\boxed{\text{Area} = \frac{1}{m^2+1}\int_p^q \left[f(x)-mx-b\right]\left[1+mf'(x)\right]dx}\ !$$

phew.

Now, recall that by the slicing method, the volume of $\mathcal{R}$ rotated around $y=mx+b$ is given by

$$V = \int_p^q A \, dx$$

where A is the area of a slice. A slice is given by



a circle w/ radius r (the same r we just found!).

So,

$$V = \int_p^q \pi r^2 \, du$$

$$= \pi \int_p^q \left[\frac{1}{\sqrt{m^2+1}}\left(f(x)-mx-b\right)\right]^2 \cdot \frac{1}{\sqrt{m^2+1}}\left(1+mf'(x)\right) dx$$

$$\boxed{\text{Volume} = \frac{\pi}{(m^2+1)^{3/2}} \int_p^q \left(f(x)-mx-b\right)^2\left(1+mf'(x)\right) dx} \qquad (**)$$

Use formulas (*) and (**) to complete the rest of the project.