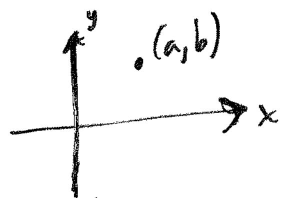
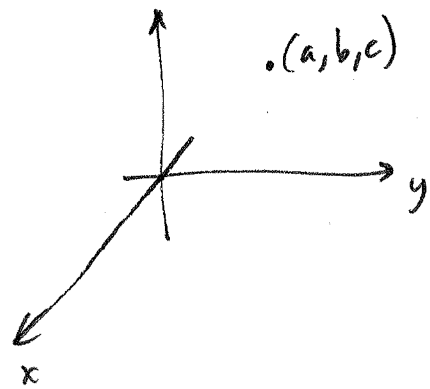
10.1: 3D coördinates

In the 2D Cartesian plane any point can be represented by an ordered pair (a,b), where a is the x-coord and b is the y-coord.



In 3D, we need an ordered triple: (a,b,c)

Standard coord axes:

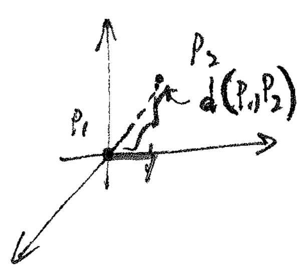


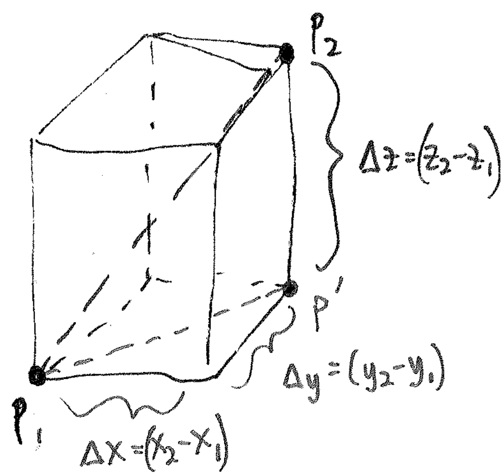
Use the corner of the room to help visualize.

The axes divide the space into "octants". The <u>first octant</u> has $x,y,z > 0$.

<u>Examples:</u> describe: $y=5$, $y=x$

<u>3D Distance formula:</u>





$$\left.\begin{array}{l} P_1=(x_1,y_1,z_1) \\ P'=(x_2,y_2,z_1) \\ P_2=(x_2,y_2,z_2) \end{array}\right\}$$

using Pythagorean Thm:

$$\boxed{d(P_1,P_2)=\sqrt{d(P_1,P')^2+(\Delta z)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Example. Let $P=(x,y,z)$ be a "variable point" and $P_0=(x_0,y_0,z_0)$ be a fixed point. Describe this set:

$$d(P, P_0) = r$$

Eq'n of a sphere: An eq'n of a sphere centered at $C=(h,k,l)$ w/ radius r is:

$$\boxed{(x-h)^2+(y-k)^2+(z-l)^2=r^2}$$

Ex. Make the center = origin.

Ex. What region is represented by $1 \le x^2+y^2+z^2 \le 4$? What is its volume?

$$\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3 = \frac{32}{3}\pi - \frac{4}{3}\pi = \boxed{\frac{28}{3}\pi \text{ units}^3}$$

Ex. (14.) Find the center and radius of the sphere:

$$x^2+y^2+z^2=4x-2y$$

$$x^2-4x+4 \quad + y^2+2y+1 \quad +z^2 = 4+1$$

$$(x-2)^2+(y+1)^2+z^2=5$$

$$\boxed{C=(2,-1,0) \qquad r=\sqrt{5}}$$

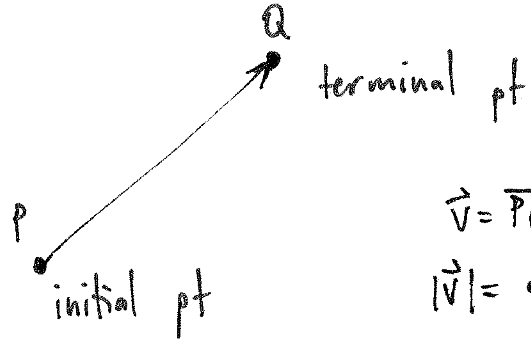
Ex. (20) Find eq'n of largest sphere w/ center $C(5,4,9)$ contained in the first octant:

$$\boxed{(x-5)^2+(y-4)^2+(z-9)^2=16}$$

10.2: Vectors

In engineering and many sciences:
A vector is a quantity that has both magnitude and direction

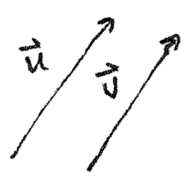
Ex. displacement, velocity, acceleration, force.



$\vec{v} = \overrightarrow{PQ}$

$|\vec{v}| = d(P,Q)$ = magnitude (or length) of $\vec{v}$.

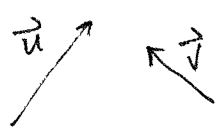
Two vectors w/ the same length and direction are "the same".
The initial and terminal pts don't really matter.



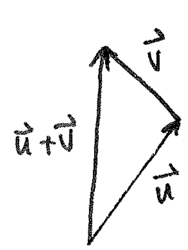
$\vec{u} = \vec{v}$

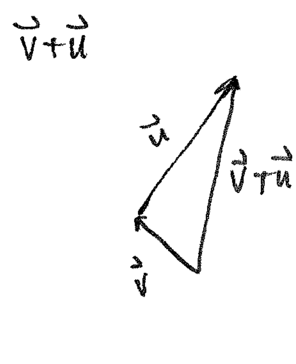
Because of this, we have a property called parallel translation, or "pick-and-plop." We can move vectors around as we please, as long as we don't change the length or direction.

Vector Addition: "tip-to-tail" * [Triangle & Parallelogram Laws]



: $\vec{u} + \vec{v} =$





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$\vec{u}+\vec{v} = \vec{v}+\vec{u}$: Commutative!

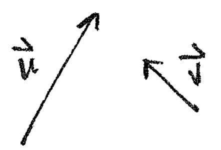
Scalar Multiplication: Let $c > 0$, and $\vec{v}$ a vector. Then $c\vec{v}$ is a vector in the same direction as $\vec{v}$, but with length $c|\vec{v}|$.

If $c = -1$, then $c\vec{v}$ switches the tip and tail.

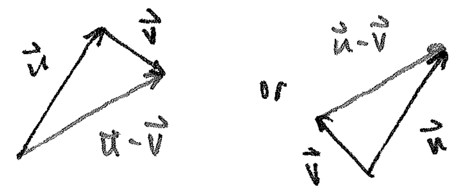
So for $c < 0$, $c\vec{v}$ is in the "opposite" direction, w/ length $|c||\vec{v}|$.

If $c = 0$ or $\vec{v} = \vec{0}$, then $c \cdot \vec{v} = \vec{0}$.

Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ Not commutative!



$\vec{u} - \vec{v} =$

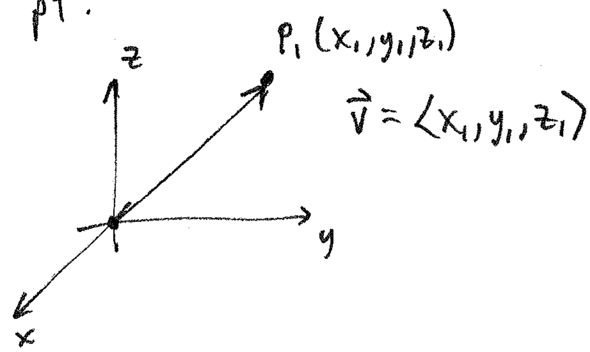


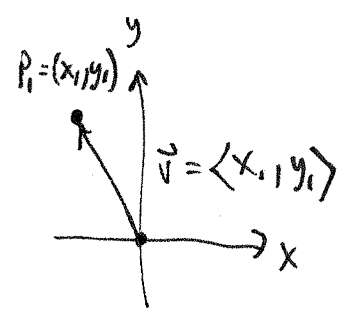
Ex. $\vec{u} - 2\vec{v}$, $\vec{v} - \vec{u}$.

(*) Notice, this was all done w/out coordinates!

Components and Standard Position:

We can put the initial pt of every vector at the origin (bc of parallel translation), then a vector is just described by its terminal pt.





We call the space of all 3-vectors $\mathbb{V}^3$, and all 2-vectors $\mathbb{V}^2$.

n-vectors: $\mathbb{V}^n$

Given initial and terminal pts only:

$A = (a_1, a_2, a_3)$
$B = (b_1, b_2, b_3)$
$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

Component form is terminal minus initial.

We can write $\vec{v} = \vec{AB}$, if we want, $\vec{v} \in \mathbb{V}^3$.

Ex. $A = (2,-3,4)$, $B = (-2,1,1)$, Find $\vec{AB} = \vec{v}$.

Def'n. If $\vec{v} \in \mathbb{V}^3$ is in standard position; i.e., $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in comp. form,

then $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the length of $\vec{v}$.

For $\vec{v} \in \mathbb{V}^n$, $|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$. Also, $\vec{v} \in \mathbb{V}^2$.

Addition, Subtraction, and Scalar Mult. in comp. form:

We'll write the formulas in $\mathbb{V}^2$.

$\vec{a} = \langle a_1, a_2 \rangle$ $\quad \vec{b} = \langle b_1, b_2 \rangle$

$$\begin{cases} \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \\ \vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle \\ c\vec{a} = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle \end{cases}$$

Exc. Write the formulae for $\mathbb{V}^n$.

Ex. $\vec{a} = \langle 4,0,3 \rangle$, $\vec{b} = \langle -2,1,5 \rangle$ $\quad$ Find: $|\vec{a}|$, $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $3\vec{b}$, $2\vec{a}+5\vec{b}$.

Properties of Vectors: Let $\vec{a}, \vec{b}, \vec{c} \in V^n$, $c, d \in \mathbb{R}$, then

1. $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ addition is commutative
2. $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$ add. is ass.
3. $\vec{a}+\vec{0} = \vec{a}$ $\vec{0}$ is add. identity
4. $\vec{a}+(-\vec{a}) = \vec{0}$ vectors have add. inverses
5. $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$ scal. mult. dist. over add.
6. $(c+d)\vec{a} = c\vec{a}+d\vec{a}$ scalar mult. dist over scalar add.
7. $(cd)\vec{a} = c(d\vec{a})$ scal. mult. is ass.
8. $1 \cdot \vec{a} = \vec{a}$ $1 \in \mathbb{R}$ is scal. mult. id.

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* Exc. Try to prove (or at least verify) these using component form. Try it in V^2 or V^3 first if V^n is too confusing.

→ Verification of (2) using Parallelogram Law in book (p. 527). ~~We already did it.~~ We did (1)

Yet another way:

Standard basis vectors:

In V^3, let $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$.

We can write any other vector in V^3 as a linear combination of these vectors.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

Then $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Ex. $\vec{a} = \langle 5,-3,2 \rangle$

$\vec{a} = 5\vec{i} - 3\vec{j} + 2\vec{k}$.

Ex. (16) $\vec{a} = 2\vec{i} - 4\vec{j} + 4\vec{k}$

$\vec{b} = 2\vec{j} - \vec{k}$

Find: $\vec{a}+\vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$, $|\vec{a}-\vec{b}|$.

Ex. (18) A unit vector is a vector of length 1.

- If $\vec{v} = \langle -2,4,2 \rangle$, find a unit vector in the direction of $\vec{v}$.
- Find a vector of length 6 in the direction of $\vec{v}$.

Ex. (19)-ish If $\vec{u} \in V^2$ is in quadrant III and makes a $\pi/4$ angle w/ the negative x-axis, find its comp. form.

Note: Any vector in V^2 can be written as

$\vec{v} = r \langle \cos\theta, \sin\theta \rangle$ where $r = |\vec{v}|$ and θ is ∡ from standard pos.

10.3: The Dot Product

Def'n. Let $\vec{a}, \vec{b} \in V^3$. i.e., $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$ ← A number!

* Also called scalar product or inner product

Defn. In V^n, $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \dots + a_nb_n$.

Ex. $\langle 2,4\rangle\cdot\langle 3,-1\rangle = 2(3) + 4(-1) = 6-4 = 2$

$\langle -1,7,4\rangle\cdot\langle 6,2,-\tfrac{1}{2}\rangle = -6 + 14 - 2 = 6$

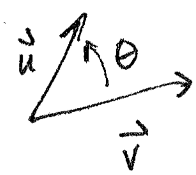
$(\vec{i}+2\vec{j}-3\vec{k})\cdot(2\vec{j}-\vec{k}) = \langle 1,2,-3\rangle\cdot\langle 0,2,-1\rangle = 0+4+3 = 7.$

Th. Properties of the dot product. If $\vec{a},\vec{b},\vec{c} \in V^n$ and $k \in \mathbb{R}$, then

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ $\qquad$ $\cdot$ is comm
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ $\qquad$ $\cdot$ distributes over $+$
4. $(k\vec{a})\cdot\vec{b} = \vec{a}\cdot(k\vec{b}) = k(\vec{a}\cdot\vec{b})$
5. $\vec{0}\cdot\vec{a} = 0$

Prove some.

Geometrically, $\cdot$ is related to the angle between two vectors.



Th. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

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$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

See book for proof.

Ex. If $|\vec{a}|=4$, $|\vec{b}|=6$, $\theta=\pi/3$, find $\vec{a}\cdot\vec{b}$. 12.

$$\vec{a}\cdot\vec{b}=4\cdot 6\cdot\cos(\pi/3)=24\cdot\left(\tfrac{1}{2}\right)=12.$$

Ex. Find the angle between $\vec{a}=\langle 2,2,-1\rangle$ and $\vec{b}=\langle 5,-3,2\rangle$.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta$$

$$\Rightarrow \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$\left.\begin{aligned}\vec{a}\cdot\vec{b}&=\langle 2,2,-1\rangle\cdot\langle 5,-3,2\rangle=10-6-2=2\\ |\vec{a}|&=\sqrt{4+4+1}=3\\ |\vec{b}|&=\sqrt{25+9+4}=\sqrt{38}\end{aligned}\right\}\quad \cos\theta=\frac{2}{3\sqrt{38}}$$

$$\text{so}\quad \theta=\cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)\approx 1.46$$

Ex. Find the angle between $\vec{u}=2\vec{i}+2\vec{j}-\vec{k}$

$\vec{v}=5\vec{i}-4\vec{j}+2\vec{k}$

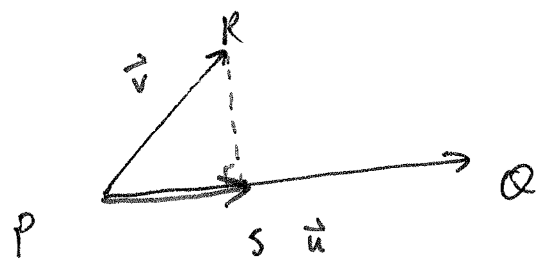
$$\left.\begin{aligned}\vec{u}\cdot\vec{v}&=\langle 2,2,-1\rangle\cdot\langle 5,-4,2\rangle=10-8-2=0\\ |\vec{u}|&=3\\ |\vec{v}|&=\sqrt{25+16+4}=\sqrt{45}\end{aligned}\right\}\quad \begin{aligned}\cos\theta&=\frac{0}{3\sqrt{45}}\\ \cos\theta&=0\\ \theta=\cos^{-1}(0)&=\frac{\pi}{2}\\ &\Rightarrow \vec{u}\perp\vec{v}.\end{aligned}$$

Th. Two vectors $\vec{a}, \vec{b} \in V^n$ are orthogonal iff $\vec{a}\cdot\vec{b} = 0$.

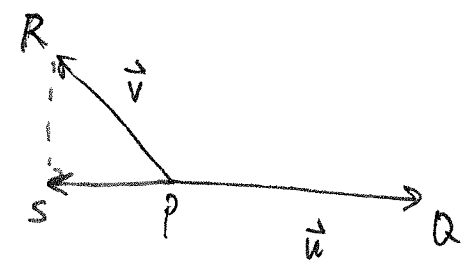
More: if $\vec{a}\cdot\vec{b} > 0$, then $0 \le \theta < \frac{\pi}{2}$

if $\vec{a}\cdot\vec{b} < 0$, then $\frac{\pi}{2} < \theta \le \pi$

Projections:



$\vec{PS} = \text{proj}_{\vec{u}}\vec{v}$



$\vec{PS} = \text{proj}_{\vec{u}}\vec{v}$.

Notice also: $\vec{PR} = \vec{PS} + \vec{SR}$

(*) projection of $\vec{v}$ onto $\vec{u}$.

where $\vec{SR} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \text{orth}_{\vec{u}}\vec{v}$

and $\vec{PS} \perp \vec{SR}$

So we can decompose $\vec{PR}$ into orthogonal components w/ one in the direction of $\vec{PQ}$.

Defn. $\vec{u}, \vec{v} \in V^n$.

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}}_{\#}\vec{u}$$

Ex. $\vec{u} = \langle -2, 3, 1 \rangle \qquad \vec{v} = \langle 1, 1, 2 \rangle$

$$\text{proj}_{\vec{u}}\, \vec{v} = ? = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$\vec{u}\cdot\vec{v} = \langle -2,3,1\rangle \cdot \langle 1,1,2\rangle = -2+3+2 = 3$$

$$\vec{u}\cdot\vec{u} = \langle -2,3,1\rangle \cdot \langle -2,3,1\rangle = 4+9+1 = 14$$

$$\text{proj}_{\vec{u}}\, \vec{v} = \frac{3}{14}\langle -2, 3, 1\rangle = \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

The scalar projection of $\vec{u}$ on $\vec{v}$ is the ~~length~~ of ~~coefficient~~ length and direction $\text{proj}_{\vec{u}}\, \vec{v}$.

Defn. $\vec{u}, \vec{v} \in V^n$. $\quad \text{comp}_{\vec{u}}\, \vec{v} = \left| \| \text{proj}_{\vec{u}}\, \vec{v} \| \right| = \left| \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\,\vec{u} \right| = \left| \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} \right| \left| \frac{\vec{u}}{|\vec{u}|} \right| = \left| \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} \right|$

$$\text{comp}_{\vec{u}}\, \vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$$

Ex. (24) $\vec{a} = \langle 1,2 \rangle \quad \vec{b} = \langle -4, 1\rangle \qquad$ Find $\text{comp}_{\vec{a}}\, \vec{b}$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{-4+2}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

Prove: $\text{orth}_{\vec{u}}\, \vec{v} \perp \vec{u}$ and

&

$\text{proj}_{\vec{u}}\, \vec{v}$.

Ex (43). Cauchy - Schwarz Inequality: $\vec{u}, \vec{v} \in \mathbb{V}^n$

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$$

Ex (44). Triangle Inequality : $\vec{u}, \vec{v} \in \mathbb{V}^n$

$$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$$

pf.
$$\begin{aligned} |\vec{u}+\vec{v}|^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot\vec{v} \\ &\leq (|\vec{u}| + |\vec{v}|)^2 \end{aligned}$$

$$\Rightarrow \quad |\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}| \quad \square$$

Ex (45). Parallelogram Law: $\vec{u}, \vec{v} \in \mathbb{V}^n$

$$|\vec{u}+\vec{v}|^2 + |\vec{u}-\vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$$

10.4: The Cross Product

Def'n. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

Or it can be done using determinants.

Ex. $\vec{a} = \langle 1, 3, 4 \rangle$ $\vec{b} = \langle 2, 7, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

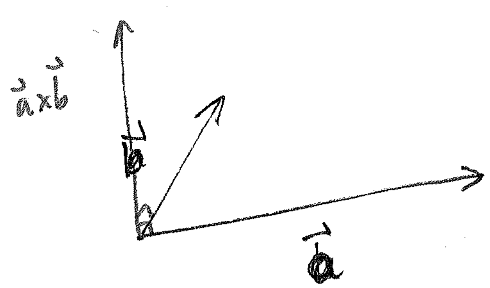
$$= (-15-28)\vec{i} - (-5-8)\vec{j} + (7-6)\vec{k}$$

$$= -43\vec{i} + 13\vec{j} + \vec{k} = \langle -43, 13, 1 \rangle$$

FTIS: $\vec{a} \times \vec{a} = \vec{0}$ for any $\vec{a} \in V^3$.

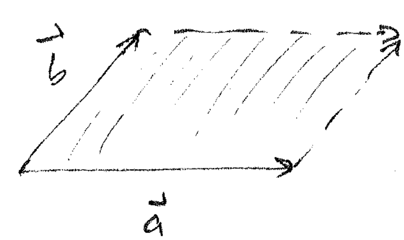
Geometrically:

direction



Right hand rule

length: $|\vec{a} \times \vec{b}|$ = area of parallelogram



Ex. (26) Find a nonzero vector orthogonal to the plane containing $P(2,1,5)$, $Q(-1,3,4)$, $R(3,0,6)$

$$\overrightarrow{PQ} = \langle 3,-2,1\rangle$$

$$\overrightarrow{QR} = \langle -4,3,-2\rangle$$

$$\overrightarrow{PQ}\times\overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -4 & 3 & -2 \end{vmatrix} = \langle 1, +2, 1\rangle$$

Find the area of ΔPQR:

$$A = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right| = \frac{1}{2}\sqrt{1+4+1} = \boxed{\frac{1}{2}\sqrt{6}}$$

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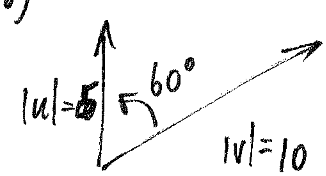
Th. If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

Ex. (10)



Find $|\vec{u}\times\vec{v}|$

$$= 5\cdot 10\cdot \sin\left(\frac{\pi}{3}\right) = 50\left(\frac{\sqrt{3}}{2}\right) = 25\sqrt{3}$$

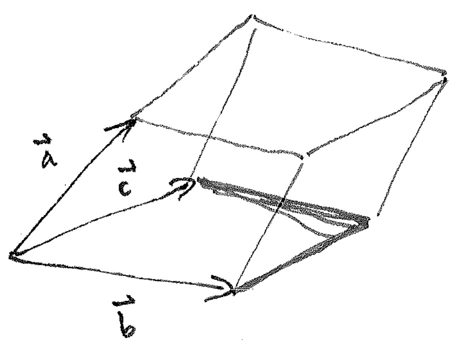
$60° = \frac{\pi}{3}$ rad

Cor. $\vec{a},\vec{b} \in V^3 \neq 0$ $\quad \vec{a}\parallel\vec{b} \Leftrightarrow \vec{a}\times\vec{b} = \vec{0}$ $\quad\square$

Properties: $\vec{a}, \vec{b}, \vec{c} \in V^3$ $\quad c \in \mathbb{R}$

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ "anti ~~symmetric~~ commutative"
2. $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ "Scalar Triple Product"
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

The triple product has geometric meaning:



Parallelépiped

The volume of this parallelepiped is

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Ex. (30) $\vec{a} = \vec{i} + \vec{j} - \vec{k} = \langle 1, 1, -1 \rangle$ Find vol. of parallelepiped.

$\vec{b} = \vec{i} - \vec{j} + \vec{k} = \langle 1, -1, 1 \rangle$

$\vec{c} = -\vec{i} + \vec{j} + k = \langle -1, 1, 1 \rangle$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(-2) - 1(2) - 1(0)$$

$$= -2 - 2$$

$$= -4$$

so, volume $= 4$.