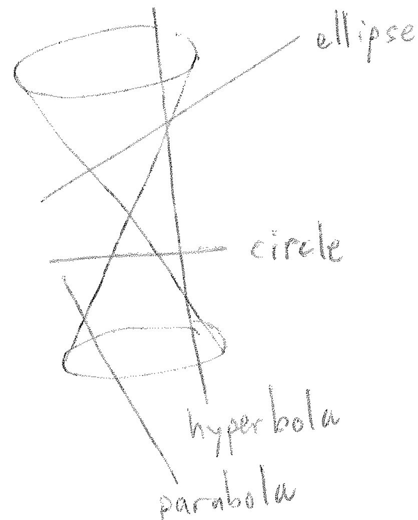
§9.5: Conic Sections

Review handout:

4 types of conic sections



These all have Cartesian equations. (See Project.): if $(h,k)=(0,0)$

Circle: $x^2+y^2=r^2$ OR $\boxed{\dfrac{x^2}{r^2}+\dfrac{y^2}{r^2}=1}$

"Locus of points" definitions...

Parabola: $\boxed{y=ax^2}$

Ellipse: $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$, $a\geq b>0 \Rightarrow$ horizontal

Hyperbola: $\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$

Exc. Write these out for $(h,k)\neq(0,0)$.

Our job today is to write these in Polar coordinates.

Thm. Let F be a fixed point (called the <u>focus</u>) and ℓ be a fixed line (called the <u>directrix</u>) in a plane. Let e be a fixed positive number (called the <u>eccentricity</u>). The set of all points in the plane such that

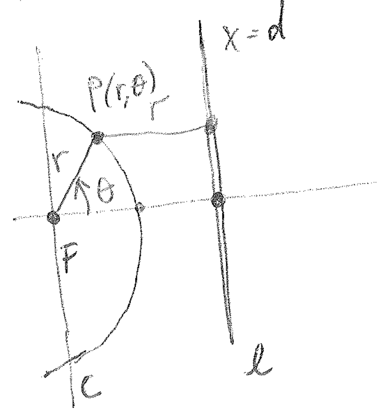
$$\frac{d(P,F)}{d(P,\ell)} = e$$

is a conic section. The conic is

i.) an ellipse if $e<1$
ii) a parabola if $e=1$
iii) a hyperbola if $e>1$

Proof. If $e=1$, then this is just the def'n of a parabola.

For the other two, we might as well put $F=(0,0)$ and make ℓ parallel to the y-axis, say $x=d$.



$P(r,\theta)_p = P(r\cos\theta, r\sin\theta)_c$

Put $d(P,F)=r$ so that $d(P,\ell)=d(P,Q)= d-r\cos\theta$

Then $$\frac{d(P,F)}{d(P,\ell)} = \frac{r}{d-r\cos\theta} = e$$

so $$r = e(d-r\cos\theta) \quad (*)$$

Square both sides of (*):

$$r^2 = e^2(d - r\cos\theta)^2 = e^2(d-x)^2 = e^2d^2 - 2e^2dx + e^2x^2$$

$$r^2 = e^2d^2 - 2e^2d\,r\cos\theta + r^2\cos^2\theta$$

$$\Rightarrow \quad x^2 + y^2 = e^2d^2 - 2e^2dx + e^2x^2$$

$$\underbrace{(1-e^2)x^2 + 2e^2dx}_{\text{complete the square!}} + y^2 = e^2d^2$$

$$x^2 + \frac{2e^2d}{1-e^2}x + \frac{e^4d^2}{(1-e^2)^2} + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2} + \frac{e^4d^2}{(1-e^2)^2}$$

$$b = \frac{2e^2d}{1-e^2}$$

$$\frac{b}{2} = \frac{e^2d}{1-e^2}$$

$$\left(\frac{b}{2}\right)^2 = \frac{e^4d^2}{(1-e^2)^2}$$

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{(1-\cancel{e^2})e^2d^2 + \cancel{e^4d^2}}{(1-e^2)^2}$$

$$\Rightarrow \quad \left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

multiply through by $\frac{(1-e^2)^2}{e^2d^2}$. (or divide by $\frac{e^2d^2}{(1-e^2)^2}$) to get

Suppose $e^2 < 1$ here.

$$\frac{\left(x + \frac{e^2d}{1-e^2}\right)^2}{\frac{e^2d^2}{(1-e^2)^2}} + \frac{y^2}{\frac{e^2d^2}{(1-e^2)}} = 1$$

If we put $a^2 = \frac{e^2d^2}{(1-e^2)^2}$, "$b^2$" $= \frac{e^2d^2}{1-e^2}$, $h = \frac{-e^2d}{1-e^2}$, then

this is

$$\frac{(x-h)^2}{a^2} \text{ "+" } \frac{y^2}{b^2} = 1 \quad \text{when } e < 1$$

and

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{when } e > 1.$$

In Cartesian coord's, eccentricity of an ellipse is given by $e = c/a$, where c is the distance of the focus from the center.

By right Δ trig, $c^2 = a^2 - b^2$

$$= \frac{e^2d^2}{(1-e^2)^2} - \frac{e^2d^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2} - \frac{(1-e^2)e^2d^2}{(1-e^2)^2}$$

$$= \frac{e^4d^2}{(1-e^2)^2}$$

so $c = \dfrac{e^2d}{1-e^2} = -h$ ✓

and $e = \dfrac{e^2d/1-e^2}{ed/1-e^2} = \dfrac{e^2d}{ed} = e$ ✓.

If $e<1$, get similar results for hyperbola.

For a hyperbola $c^2 = a^2 + b^2$.

We can use eq'n (*) to write polar equations for ellipses and hyperbolas:

$$(*) \quad r = e(d - r\cos\theta)$$

Solve for r: $r = ed - re\cos\theta$

$$r + re\cos\theta = ed$$

$$\boxed{r = \frac{ed}{1+e\cos\theta}} \quad \begin{cases} \text{parabola if } e=1 \\ \text{ellipse if } e<1 \\ \text{hyperbola if } e>1 \end{cases}$$

Also, we can replace $\cos\theta$ w/ $\sin\theta$ to get

$$\boxed{r = \frac{ed}{1+e\sin\theta}}$$ same e restrictions.

__Ex__. Find a polar equation for a parabola that has its focus at the origin and whose directrix is $y = -6$.

__Sol'n__: $e = 1$, $d = 6$, which of the eqns to use?

Use Wolfram to plot them all.

See that

$$r = \frac{6}{1 - \sin\theta}$$

is the one.

directrix determines $\pm(\sin/\cos)\,\theta$

__Ex__. $$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, locate directrix, find other focus.

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta} = \frac{\overbrace{\tfrac{2}{3}}^{e}\,\overbrace{5}^{d}}{1 - \frac{2}{3}\cos\theta}$$

$e = \frac{2}{3}$, directrix: $x = -5$.

vertices when $\theta = 0, \pi$

$$r(0) = \frac{10}{3 - 2(1)} = \frac{10}{1} = 10$$

$$r(\pi) = \frac{10}{3 - 2(-1)} = \frac{10}{5} = 2$$

$(4, 0)$ center.

other focus at $(8, 0)$!

Ex. $r = \dfrac{12}{2+4\sin\theta}$ hyperbola