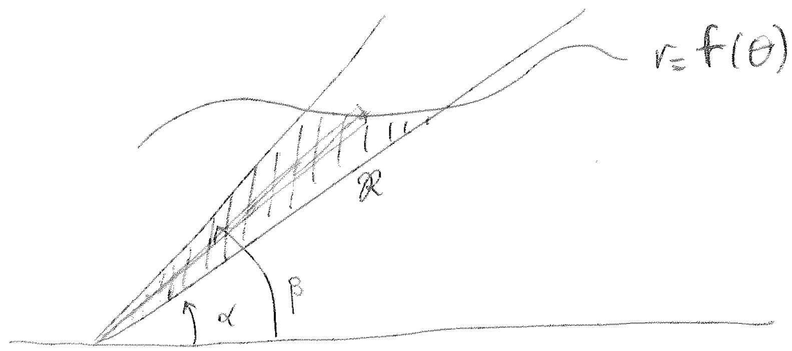
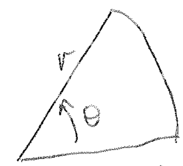
Areas and lengths of Polar curves:

$r = f(\theta) = r(\theta)$



recall



$A = \frac{1}{2} r^2 \theta$

Area of a sector of a circle.

so Area of $\mathcal{R}$ is approximately:

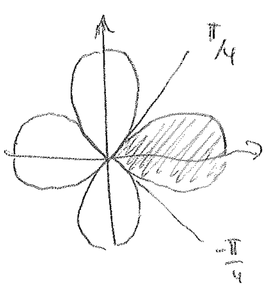
$$A \approx \sum_{n=1}^{m} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta \qquad \text{"Riemann Sum"}$$

and the actual area is given by:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta = \boxed{\int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 d\theta}$$

Ex. Find the area of on petal of the 4-leaved rose:

$r = \cos 2\theta$



$$A = \int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta = 2 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

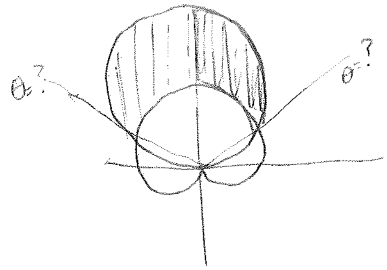
$$= \frac{1}{2} \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{4} \cdot \frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{8}}$$

Ex. Find the area inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1 + \sin\theta$.



$$3\sin\theta = 1 + \sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$$
$$A = \frac{1}{2}\int_\alpha^\beta r_o^2 - r_i^2 \, d\theta$$

$$\cancel{A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta - 1 - \sin\theta)^2 \, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta - 1)^2 \, d\theta}$$

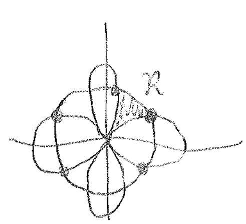
$$\cancel{= \int_{\pi/6}^{5\pi/6} 4\sin^2\theta - 4\sin\theta + 1 \, d\theta}$$

$$r_o = 3\sin\theta \Rightarrow r_o^2 = 9\sin^2\theta$$
$$r_i = 1 + \sin\theta \Rightarrow r_i^2 = 1 + 2\sin\theta + \sin^2\theta$$
$$r_o^2 - r_i^2 = 8\sin^2\theta - 2\sin\theta - 1$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\sin\theta - 1 \, d\theta$$
$$= \int_{\pi/6}^{\pi/2} -4\cos 2\theta - 2\sin\theta + 3 \, d\theta$$
$$= 3\theta + 2\cos\theta - 2\sin 2\theta \Big|_{\pi/6}^{\pi/2}$$
$$= \frac{3\pi}{2} - \frac{3\pi}{6} + 2\cos\frac{\pi}{2} - 2\cos\frac{\pi}{6} - 2\sin\pi + 2\sin\frac{\pi}{3}$$
$$= \pi + 0 - \cancel{\sqrt{3}} + 0 + \cancel{\sqrt{3}} = \boxed{\pi}$$

phew.

Ex. Find all points on the intersection of $r = \cos 2\theta$ and $r = \frac{1}{2}$, then find A of R.



$$\cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

but this only gives 4 of the 8 intersections.

There are 4 other intersection points: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.
These come from setting $\cos 2\theta = -\frac{1}{2}$.
We want

$$A(\mathcal{R}) = \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/3} \frac{1}{4} - \cos^2 2\theta \; d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{4}\theta\Big|_{\pi/6}^{\pi/3} - \frac{1}{4}\int_{\frac{\pi}{6}}^{\pi/3} 1 + \cos 4\theta \; d\theta$$

$$= \frac{1}{8}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{8}\left(\frac{\pi}{6}\right) - \frac{1}{4}\left(\frac{\pi}{6}\right) - \frac{1}{16}\left(\sin\frac{4\pi}{3} - \sin\frac{4\pi}{6}\right)$$

$$= -\frac{\pi}{48} - \frac{1}{16}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{3\sqrt{3}-\pi}{48}}$$

Arc Length:

Same trick as we used w/ derivates (slope of tan lines)

$$\left.\begin{aligned} x &= r(\theta)\cos\theta \\ y &= r(\theta)\sin\theta \end{aligned}\right\} \text{ both parametrized by } \theta$$

recall the parametrized formula for arc length

$$L_\gamma(a,b) = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\; d\theta$$

$$\dot{x} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \Rightarrow (\dot{x})^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$\dot{y} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \Rightarrow (\dot{y})^2 = \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$+$

$$\left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

So arc length becomes:

ehh, not a parameter for r.

$$L_\gamma(a,b) = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + (\dot{r})^2}\, d\theta$$

<u>Ex</u>. Find the perimeter of the cardioid: $r = 1 + \sin\theta$

$$r^2 = 1 + 2\sin\theta + \sin^2\theta$$

$$\frac{dr}{d\theta} = \cos\theta \qquad \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + 2\sin\theta + \sin^2\theta + \cos^2\theta = 2 + 2\sin\theta$$

$$L(0, 2\pi) = \int_0^{2\pi} \sqrt{2(1+\sin\theta)}\, d\theta$$

$$= \int_0^{2\pi} \frac{2(1+\sin\theta)}{\sqrt{2+2\sin\theta}}\, d\theta \quad \text{ew!}$$

Wolfram|Alpha: $\boxed{L = 8}$.

Hand out College Algebra Conic Section worksheet.