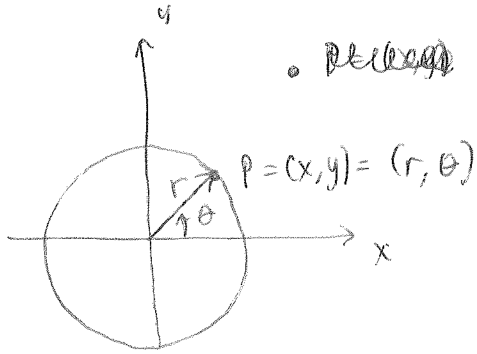
§9.3: Polar Coordinates

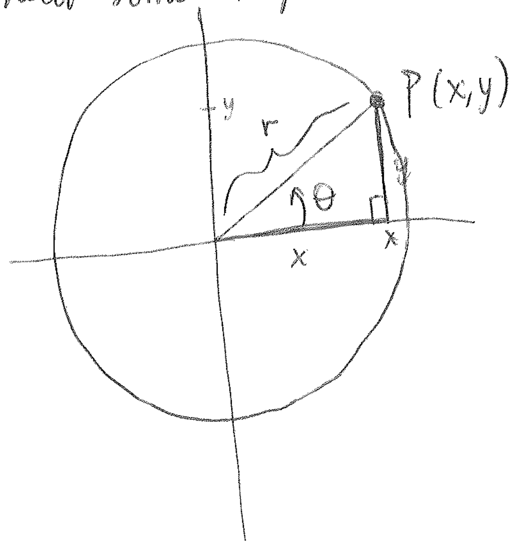


Any pt in $\mathbb{R}^2$ can be described by the radius of the circle that it lies on, and the angle that its "position vector" makes w/ x^+-axis.

$$P=(r,\theta).$$

P has (x,y)-coords and (r,θ)-coords. We should be able to write one in terms of the other (i.e., swith between them).

To do so, we need some trig:



Pythagorean Thm: $r^2 = x^2+y^2 \Rightarrow \boxed{r=\sqrt{x^2+y^2}} \geq 0$

$$r=0 \Leftrightarrow P=(0,0)$$

Right Δ trig: $\tan\theta = \frac{O}{A} = \frac{y}{x} \Rightarrow \boxed{\theta = \arctan\left(\frac{y}{x}\right)}$

These formulae turn (x,y) into (r,θ) (or Cartesian $\to$ Polar)

$$P(x,y)_c = \left(\sqrt{x^2+y^2},\ \arctan\left(\frac{y}{x}\right)\right)_p$$

Also by Right Δ trig:

$\cos\theta = \frac{x}{r}$, so $\boxed{x = r\cos\theta}$

and $\sin\theta = \frac{y}{r}$, so $\boxed{y = r\sin\theta}$

(*) These formulae change Polar to Cartesian:

$$C(r,\theta)_p = (r\cos\theta, r\sin\theta)_c$$

$P(C(r,\theta)_p) = (r,\theta)_p$, and
$C(P(x,y)_c) = (x,y)_c$

so P and C are inverse transformations.

Ex. Convert $(2, \pi/3)$ from polar to Cartesian.

$$x = r\cos\theta = 2\cos(\pi/3) = 2(1/2) = 1$$
$$y = r\sin\theta = 2\sin(\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$$

so $C(2, \pi/3)_p = \boxed{(1, \sqrt{3})_c}$

Ex. Convert $(-1, 1)$ from Cartesian to polar:

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

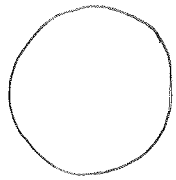
$$\theta = \arctan\left(\frac{1}{-1}\right) = -\pi/4$$

so, $P(-1,1)_c = \boxed{(\sqrt{2}, -\pi/4)_p}$

Polar Curves.

the graph of a polar equation $r = f(\theta)$ (or $F(r,\theta) = 0$) (r,θ)
consists of all points who have at least one polar rep. that satisfies the eqn.

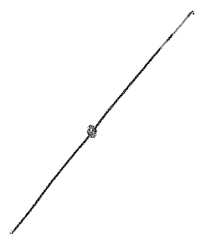
Ex. $r = 2$



circle w/ radius 2.

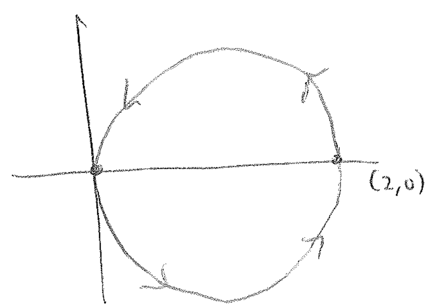
Ex. $\theta = 1$

line w/ angle $\theta = 1$ (slope = $\tan 1$)



Ex. $r = 2\cos\theta$

θ	$r = 2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\sqrt{2}$
$\pi/3$	1
$\frac{\pi}{2}$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	-2



Write in Cartesian coords:

$$r = 2\frac{x}{r}$$

$$r^2 = 2x$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

circle w/ center $(1,0)$ and $r = 1$.

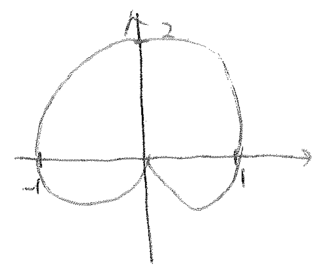
Ex. The Cardiod.

$$r = 1 + \sin\theta$$

$\sin\theta$: $0 \to \frac{\pi}{2} \to \pi \to \frac{3\pi}{2} \to 2\pi$

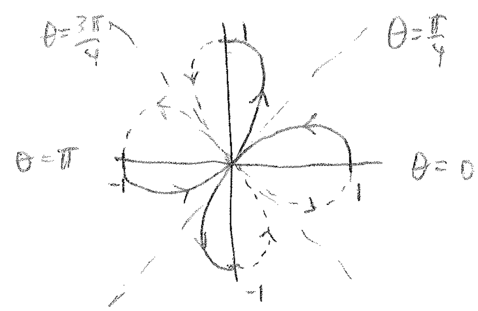
$0 \xrightarrow{\text{inc}} 1 \xrightarrow{\text{dec}} 0 \xrightarrow{\text{dec}} -1 \xrightarrow{\text{inc}} 0$

$1 + \sin\theta$: $1 \xrightarrow{\text{inc}} 2 \xrightarrow{\text{dec}} 1 \xrightarrow{\text{dec}} 0 \xrightarrow{\text{inc}}$



← horrible picture. Use computer?

Ex. Four-leaved rose $\quad r = \cos 2\theta$



Derivatives: tangents to polar curves.

Idea: regard θ as a parameter, $r = f(\theta)$, and write

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Then $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{dy/d\theta}{dx/d\theta} = \cancel{\dfrac{dr/d\theta \cos\theta - r\sin\theta}{dr/d\theta \sin\theta + r\cos\theta}}$

$$= \frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta \cos\theta - r\sin\theta} = \boxed{\frac{\dot{r}\sin\theta + r\cos\theta}{\dot{r}\cos\theta - r\sin\theta}}$$

Horizontal tangent lines occur when $\dot{y}=0$ (and $\dot{x}\neq 0$), and vertical tangents when $\dot{x}=0$ (but $\dot{y}\neq 0$).

tangents at the pole (where $r=0$) are given by $\frac{dy}{dx}=\tan\theta$ if $\dot{r}\neq 0$.

Ex. $r=\cos 2\theta$ (the ~~[illegible]~~ 4-l. rose)

$r=0$ when $\theta=\frac{\pi}{4}, \frac{3\pi}{4}$, etc.

at $\theta=\frac{\pi}{4}$, $\frac{dy}{dx}=\tan\frac{\pi}{4}=1$

$\theta=\frac{3\pi}{4}$, $\frac{dy}{dx}=\tan\frac{3\pi}{4}=-1$

this matches the picture! ☺

Ex. Find the slope of the tan line $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$ for the cardiod $r=1+\sin\theta$.

$\dot{r}=\cos\theta$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos\theta\sin\theta+(1+\sin\theta)\cos\theta}{\cos^2\theta-(1+\sin\theta)\sin\theta}$$

$$=\frac{\cos\frac{\pi}{3}\sin\frac{\pi}{3}+(1+\sin\frac{\pi}{3})\cos\frac{\pi}{3}}{(\cos\pi/3)^2-(1+\sin\frac{\pi}{3})\sin\frac{\pi}{3}}$$

$$=\frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+\left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2-\left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}=\frac{\frac{\sqrt{3}}{4}+\frac{2}{4}+\frac{\sqrt{3}}{4}}{\frac{1}{4}-\frac{2\sqrt{3}}{4}-\frac{3}{4}}$$

$$=\frac{2\sqrt{3}+2}{-2-2\sqrt{3}}=\frac{\sqrt{3}+1}{-(\sqrt{3}+1)}$$

$$=\boxed{-1}\ !$$

Ex. 55! Show that $r = a\sin\theta + b\cos\theta$, $ab \neq 0$, represents a circle. Find its center and radius.

$$r^2 = a\, r\sin\theta + b\, r\cos\theta$$

$$x^2 + y^2 = ay + bx$$

$$x^2 - bx + \left(\tfrac{b}{2}\right)^2 + y^2 - ay + \left(\tfrac{a}{2}\right)^2 = 0 + \left(\tfrac{b}{2}\right)^2 + \left(\tfrac{a}{2}\right)^2$$

$$\left(x - \tfrac{b}{2}\right)^2 + \left(y - \tfrac{a}{2}\right)^2 = \frac{b^2 + a^2}{4}$$

so $C = \left(\frac{b}{2}, \frac{a}{2}\right)$ and $r = \dfrac{\sqrt{a^2+b^2}}{2}$. WORD!

Ex. 45. Show that the curve $r = \sin\theta\tan\theta$ (Cissoid of Diocles) has a v. asymptote at $x=1$, and the curve lies entirely within the v. strip $0 \leq x < 1$.

$$r \cdot r = \frac{r\sin\theta \; r\sin\theta}{r\cos\theta}$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y - xy}$$

$$x^2 + y^2 = y \cdot \frac{y}{x}$$

$$x^2 + y^2 = \frac{y^2}{x} \longrightarrow y^2(1-x) = x^3$$

$$y = \pm\sqrt{\frac{x^3}{1-x}} \longleftarrow \text{V.A.}$$

$$\frac{d}{dx}\left[x^3 + xy^2 = y^2\right]$$

$$3x^2 + y^2 + xy\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[2y - xy\right] = 3x^2 + y^2$$

$$\frac{x^3}{1-x} \geq 0$$

$$x^3 \geq 1 - x$$

only when $0 \leq x < 1$

Sign chart.