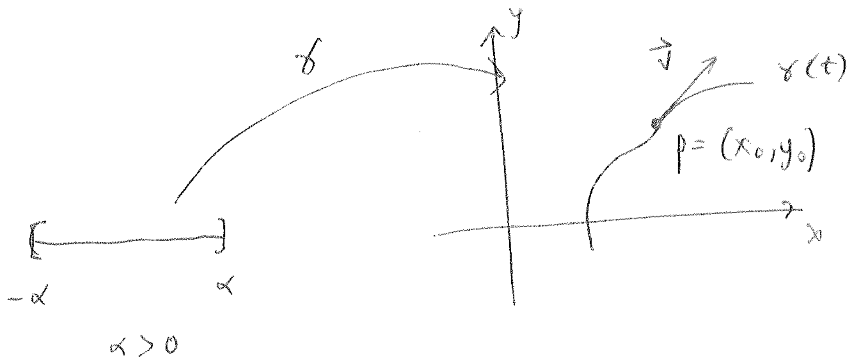
§9.2 cont'd:

Let $\gamma(t) = (x(t), y(t))$ be a parametrized curve in $\mathbb{R}^2$, and the p be a point in $\mathbb{R}^2$ that γ passes through; i.e.,



$\gamma(0) = p$
↑ position of particle

velocity of particle

$\vec{v}$ is ~~a~~ tangent vector to γ at p: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) = \vec{v}$

The direction of $\vec{v}$ is the slope of the tangent line to γ at p:

$$\frac{dy}{dx}\Big|_p = \boxed{\cancel{\dot{\gamma}(0) = \frac{d\gamma}{dt}(0)}}$$

~~where~~ $\boxed{\cancel{\dot{\gamma} = \frac{d\gamma}{dt}}} = \frac{dy}{dt} \cdot \frac{dt}{dx}\Big|_{t=0} = \frac{dy/dt}{dx/dt}\Big|_{t=0}$ when $\frac{dx}{dt} \neq 0$.

In general, at points along γ, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Second derivative is not so simple:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{dx/dt}$$

← need quotient rule!

$$= \left(\frac{dx}{dt}\right)^{-1} \cdot \frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{(dx/dt)^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

In Newtonian notation: $\boxed{\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}}$

* We usually don't need to actually use this formula b/c we already know a formula for dy/dx in terms of t. But it's a good exercise. ☺

Ex. let $\gamma(t) = (t^2, t^3 - 3t)$

so $x(t) = t^2$
$y(t) = t^3 - 3t$

Find dy/dx and d^2y/dx^2, and find the points on γ where the tangent is horizontal or vertical and the intervals of concavity.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{3t^2-3}{2t}}$$

Horizontal when $dy/dt = 0$: $3t^2 - 3 = 0$

$3(t^2-1) = 0$

$t = \pm 1$

$\gamma(1) = (1, -2)$, $\gamma(-1) = (1, 2)$

Vertical when $dx/dt = 0$: $2t = 0$

$t = 0$

$\gamma(0) = (0, 0)$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{1}{2t} \cdot \frac{2t(6t) - (3t^2-3)(2)}{(2t)^2} =$$

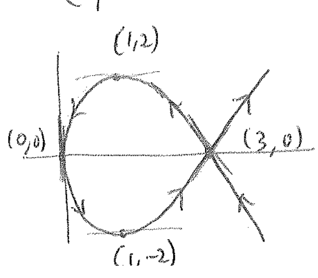
$$= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6}{8} \cdot \frac{t^2+1}{t^3} = \boxed{\frac{3}{4} \cdot \frac{t^2+1}{t^3}}$$

t^2+1 is always positive

t^3 is $\begin{cases} \text{neg for } t<0 \\ \text{pos for } t>0 \end{cases}$

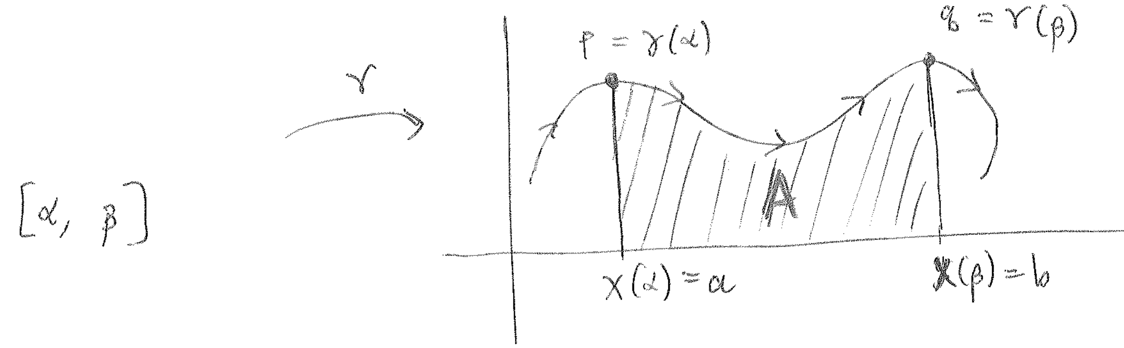
so $\gamma(t)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

graph:



Areas:

Again, let $\gamma(t) = (x(t), y(t))$ be a parametrized curve. Now assume that the parametrization is such that the curve is only traversed once as t increases, and that the "particle" moves in the positive direction, i.e.,



We know that

$$A = \int_a^b y\,dx$$

if we can write γ as a function $y = f(x)$.

By substituting, we have

$$y = y(t) \quad \text{and} \quad dx = \frac{dx}{dt}\,dt = \dot{x}(t)\,dt, \quad \text{so}$$

$$\boxed{A = \int_\alpha^\beta y(t)\,\dot{x}(t)\,dt}$$

Ex. Find the area under one "cycle" of the cycloid

$$\gamma(\theta) = (r(r-\sin\theta),\ r(1-\cos\theta))$$

$$y(\theta) = r(1-\cos\theta) \qquad \dot{x}(\theta) = r(1-\cos\theta)\,d\theta$$

$\frac{1}{2}(1+\cos 2\theta)$

So, $$A = \int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta$$

$$= r^2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = r^2(2\pi + \pi - 0) = \boxed{3\pi r^2}!$$

Arc Length:

Again, we already know how to find the length of a curve between 2 points when the curve can be written as $y = f(x)$:

$$L(a,b) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If we can paramatrize a curve γ by $\gamma(t) = (x(t), y(t))$ such that $t \in [\alpha, \beta]$ and $\dot{x}(t) = \frac{dx}{dt} > 0$ for all t (basically the same condition as for area), then

$$L\gamma(\alpha,\beta) = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

But since $dx/dt > 0$, we can move it in the $\sqrt{\cdot}$ to get:

$$L\gamma(\alpha,\beta) = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt \qquad (*)$$

Notice that this is consistent w/ the formula:

$$L = \int ds \quad \text{where} \quad ds^2 = dx^2 + dy^2$$

It turns out eq'n $(*)$ works for any parametrization as above. (gives the same answer.)

Ex. Use this method to find circumference of a circle:

$r(\theta) = (r\cos\theta, r\sin\theta) \qquad \theta \in [0, 2\pi]$

$$L\gamma(0,2\pi) = \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta = r\theta\Big|_0^{2\pi} = \boxed{2\pi r}.$$

<u>Ex.</u> Find the length of one cycle of cycloid:

$$r(\theta) = \left(r(\theta - \sin\theta),\ r(1-\cos\theta)\right), \qquad 0 \le \theta \le 2\pi$$

$$L_r(0, 2\pi) = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi} r\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\ d\theta$$

$$= \sqrt{2}\,r\int_0^{2\pi} \sqrt{1 - \cos\theta}\ d\theta$$

$$\left\{\begin{aligned} &\tfrac{1}{2}(1-\cos 2\theta) = \sin^2\theta \\ &\Rightarrow 1 - \cos\theta = 2\sin^2\left(\tfrac{\theta}{2}\right) \end{aligned}\right.$$

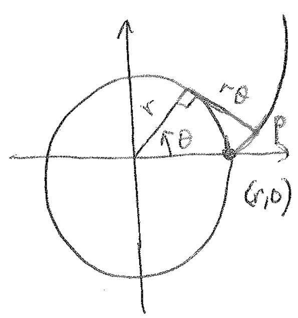
$$= \sqrt{2}\,r\int_0^{2\pi} \sqrt{2\sin^2\left(\tfrac{\theta}{2}\right)}\ d\theta$$

$$= 2r\int_0^{2\pi} \sin\left(\tfrac{\theta}{2}\right) d\theta$$

$$= -2r\cdot 2\cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi} = -4r\cos\pi + 4r\cos 0$$

$$= 4r + 4r = \boxed{8r}\ !$$

<u>Ex.</u> <u>53!</u>



← the <u>involute</u> of the circle.

show that $r(t) = \left(r(\cos\theta + \theta\sin\theta),\ r(\sin\theta - \theta\cos\theta)\right)$

* This can be an X.C. project for this section. Show all work.