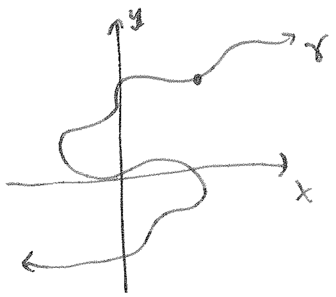
9 : Parametric Equations and Polar coörds

9.1 : Parametric Curves



γ is not a function, but at any point (x,y) we can think of
$(x(t), y(t)) = \gamma(t)$
as the position of γ at some time t, so x, y are functions of a parameter t.

the eqns $\left\{ \begin{array}{l} x = f(t) = x(t) \\ y = g(t) = y(t) \end{array} \right\}$ are called parametric eqn's.

t does not necessarily represent time, and we could use any other letter, but t is "usually" time in applications.

Ex. Sketch: $x = t^2 - 2t$
$y = t + 1$

We could make a table of values, and plot points,

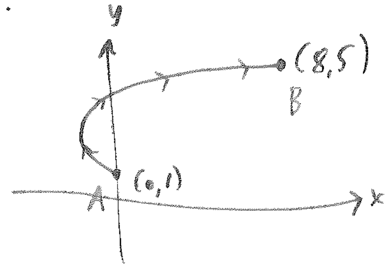
also $x = (y-1)^2 - 2(y-1)$
$= y^2 - 2y + 1 - 2y + 2$
$= y^2 - 4y + 3$

So the parametric curve $\gamma = (x(t), y(t))$ is also given by $y^2 - 4y + 3$.

Since t can be anything in this example, we get the whole parabola.

Ex. What if we restrict to $x = t^2 - 2t$
$y = t + 1$
$0 \le t \le 4$

Then we get:



If we think of r as a particle, then it "moves from A to B."

here $r(0) = (x(0), y(0)) = (0,1)$ is the <u>initial pt</u> and
$r(4) = (x(4), y(4)) = (8,5)$ is the <u>terminal pt</u>.

In general, if $r(t) = (x(t), y(t))$, $a \le t \le b$, then
$r(a)$ is initial and $r(b)$ is terminal.

<u>Ex</u>. $\left.\begin{array}{l} x = \cos t \\ y = \sin t \\ 0 \le t \le 2\pi \end{array}\right\}$ $x^2 + y^2 = 1 \Rightarrow$ <u>Unit circle!</u>

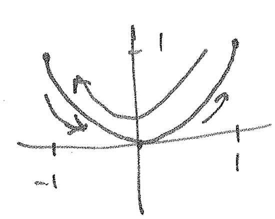
<u>Ex</u>. $\left.\begin{array}{l} x = \sin 2t \\ y = \cos 2t \\ 0 \le t \le 2\pi \end{array}\right\}$ travels the circle twice! (starting at "$\frac{\pi}{2}$")

<u>Ex</u>. Find parametric eqns for a circle centered at (h,k) w/ radius r.

$x = h + r\cos t$
$y = k + r\sin t$
$t \in [0, 2\pi]$

<u>Ex</u>. Sketch $r(t) = (x(t), y(t))$ if $x = \sin t$ and $y = \sin^2 t$.

Notice $y = x^2$ but as t varies $-1 \le x, y \le 1$

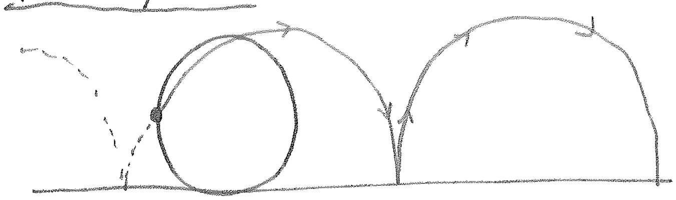


On a calculator or wolfram:

$$\left.\begin{array}{l} x = t + 2\sin 2t \\ y = t + 2\cos 5t \end{array}\right\}, \quad \left.\begin{array}{l} x = 1.5\cos t - \cos 30t \\ y = 1.5\sin t - \sin 30t \end{array}\right\}, \quad \text{and}$$

$$\left.\begin{array}{l} x = \sin(t + \cos 100t) \\ y = \cos(t + \sin 100t) \end{array}\right\}$$

The Cycloid



more "flat"

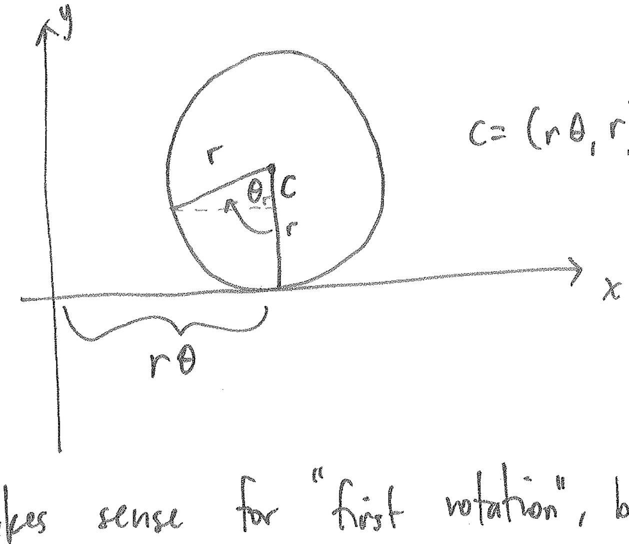
Pick a point on a "wheel" and roll the wheel. The curve that the point traces is called a <u>cycloid</u>.

Exercise for students:

Work through Example 7 on pg. 485 to derive formula $\boxed{1}$:

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \\ \theta \in \mathbb{R} \end{cases} \qquad \gamma(r,\theta) = (x(r,\theta), y(r,\theta))$$

where

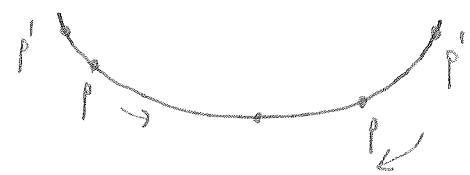


$C = (r\theta, r)$

Picture makes sense for "first rotation", but actually holds for all θ.

Christian Huygens: tautochrone problem:

No matter where a particle P is placed on an inverted cycloid, it takes the same speed to fall to the bottom.



time is the same!

Used to keep time in pendulum clocks (doesn't matter how far you swing it; always takes a "second").

Ex. (5) $x = 3t - 5$
$y = 2t + 1$

Ex. (10) $x = 4\cos\theta$
$y = 5\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Ex. (9) $x = \sin\theta$
$y = \cos\theta)$ $\quad 0 \le \theta \le \pi$

40. Swallow tail catastrophe curves

$$x = 2ct - 4t^3$$
$$y = -ct^2 + 3t^4$$

41. Lissajous figures

$$x = a \sin nt \qquad n \in \mathbb{Z}^+$$
$$y = b \cos t$$

9.2: Calculus w/ parametrized curves

We can still define tangent lines on these curves with no ambiguity.

then, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad , \text{ or}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if } dx/dt \neq 0.$$

Exc. Find $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.

Hint: it's not $\dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$!