## 8.8. Applications of Taylor Series

If $f$ has a Taylor Series expansion

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

then the $n^{th}$ partial sums are polynomial approximations.

$$T_n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $R_n(x)\to 0$ as $n\to\infty$, then $T_n(x)\to f(x)$ as $n\to\infty$.

When $n$ is large, $T_n(x)\approx f(x)$, especially near $a$.

We already did some examples w/ the partial sums.

Ex. In Einstein's theory of special relativity the mass of an object moving w/ velocity $v$ is:

$$m=\frac{m_0}{\sqrt{1-(v/c)^2}}$$

where $m_0$ is the mass of the object at rest and $c$ is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K=mc^2-m_0c^2$$

1. Show that when $v$ is very small, this $K$ agrees w/ Newtonian physics: $K=\frac{1}{2}m_0v^2$.

In particular, put $\overset{f(x)}{\overset{\parallel}{k}} = m_0 c^2\left[(1-x)^{-1/2} - 1\right]$

Find $f''(x)$:

$$f'(x) = m_0 c^2 \left(-\tfrac{1}{2}\right)(1-x)^{-3/2}$$

$$f''(x) = m_0 c^2 \left(-\tfrac{1}{2}\right)\left(\tfrac{-3}{2}\right)(1-x)^{-5/2}$$

$$= m_0 c^2 \left(\tfrac{3}{4}\right)(1-x)^{-5/2}$$

Then $$R_1(x) = \frac{3 m_0 c^2}{8(1-z)^{5/2}} \quad \frac{v^4}{c^4}$$

where $z$ is between $0$ and $\frac{-v^2}{c^2}$. We have

$c = 3 \times 10^8$ m/s and $|v| \leq 100$ m/s

So $\left|\frac{v^2}{c^2}\right| \leq \frac{10000}{9 \times 10^{16}}$

In particular,

$$R_1(x) \leq \frac{\frac{3}{8} m_0 (9 \times 10^{16})(100/c)^4}{(1 - 100^2/c^2)^{5/2}} < \underline{(4.17 \times 10^{-10})\, m_0}$$

This is a maximum bound on the error. It's very small. So this is a good approximation!