8.7. Taylor and Maclaurin Series

As we discussed on Friday: we can estimate a function by a polynomial if we insist that the function's derivatives agree w/ the polynomial's derivatives.

This leads to the idea of Taylor series.

Thm. If f has a power series expansion at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad (*)$$

Recall: In general a power series is given by:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

so this theorem gives a formula for the coefficients c_n.

Equation $(*)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots$$

is called the Taylor Series expansion of f at a.

If $a=0$, then $(*)$ becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

is called the Maclaurin Series of f.

Ex. Find the MacLaurin Series for $f(x) = e^x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\left.\begin{array}{l} f(0) = e^0 = 1 \\ f'(0) = e^0 = 1 \\ f''(0) = e^0 = 1 \end{array}\right\} \text{ so } C_n = \frac{1}{n!} \Rightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

We found on Friday that the radius of convergence is ∞.

Question: how do we know that this series actually converges to $f(x)$?

Let $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ be the n^{th} partial sum.

This is called the n^{th} Taylor polynomial of f.

Then $R_n(x) = f(x) - T_n(x)$ is the Remainder of the Taylor Series.

So, $f(x) = T_n(x) + R_n(x)$ for any n.

Therefore, a Taylor series for f converges to f iff

$$\lim_{n\to\infty} R_n(x) = 0.$$

We get:

Taylor's Formula: If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I, there is a number z strictly between x and a such that

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}}$$

Lagrange's form

Ex. Prove that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$f^{(n+1)}(x) = e^x$, so

$$R_n(x) = \frac{e^z}{(n+1)!} x^{n+1}$$

where $z \in (0, x)$.

(*) Note that z depends on x and on n.

If $x > 0$, then $0 < z < x$, so $e^z < e^x$ and

$$0 < R_n(x) = \frac{e^z}{(n+1)!} x^{n+1} < e^x \frac{x^{n+1}}{(n+1)!} \longrightarrow 0.$$

Then $R_n(x) \to 0$ by the squeeze theorem.

If $x < 0$, then $x < z < 0$, so $e^z < e^0 = 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0.$$

Therefore e^x is equal to its Taylor Series expansion. $\square$

Ex. Find the Taylor Series expansion of $f(x) = e^x$ at $x = 2$.

$f^{(n)}(x) = e^x$ so $f^{(n)}(2) = e^2$

so $\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n}$ for all x.

This is a better approx. "near" $x = 2$.

Ex. Find MacLaurin Series for $\sin x$.

$n=0$	$f(x) = \sin x$	$f(0) = 0$
$n=1$	$f'(x) = \cos x$	$f'(0) = 1$
$n=2$	$f''(x) = -\sin x$	$f''(0) = 0$
$n=3$	$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -1$
$n=4$	$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$

We get a MacLaurin Series:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

To find R, use the Remainder Thm:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, either way

$$|f^{(n+1)}(x)| \leq 1, \text{ so}$$

$$|R_n(x)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}|x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \forall x.$$

So $R = \infty$, and $\sin x$ equals its Taylor Series expansion.

Ex. Find the MacLaurin Series for $\cos x$.

$$\cos x = \frac{d}{dx}\left[\sin x\right] = \frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots\right]$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\cancel{(2n+1)}\, x^{2n}}{\cancel{(2n+1)}\,(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - + \cdots$$

Ex. Find MacLaurin Series for $x\cos x$.

Ex. Write e^{-x^2} as a power series.

1. Write MacLaurin
2. Integrate.

Ex. $f(x) = x^3$ around $a = -1$

Ex. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

Ex. Binomial Series.

Find MacLaurin series for $f(x) = (1+x)^k$

$f(x) = (1+x)^k$ $\qquad$ $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad$ $f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-2}$ $\qquad$ $f'''(0) = k(k-1)(k-2)$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n+1}$ $\qquad$ $f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

So M.S. is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Fact: ratio test: $R = 1$.

$$= \sum_{n=0}^{\infty} \underbrace{\frac{k!}{(k-n)!\,n!}}_{\binom{k}{n}} x^n$$

"k choose n" – binomial coefficients.

So the binomial series is given by

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

for any $k \in R$, and $|x| < 1$.

Ex. Find M.S. for $f(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-1/2} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}$

$$= \frac{1}{2}\left(1 - \left(-\frac{x}{4}\right)\right)^{-1/2}$$

So $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n}$ $\qquad$ $\boxed{R = 4}$

Ex. Evaluate $\int e^{-x^2}\,dx$ as a power series

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

so $$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)\, n!} + C}$$

Ex. 54. Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ using taylor series.

First: write the first 3 terms of $\tan x$ expansion.

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots}$$

$$\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - + \cdots & x - \frac{x^3}{6} + \frac{x^5}{120} - + \cdots \\
 & x - \frac{x^3}{2} + \frac{x^5}{24} - + \cdots \\
\hline
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 + - \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\hline
 & \frac{2}{15}x^5 + \cdots
\end{array}$$

so $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

and $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \boxed{\frac{1}{3}}$$

FTIS: Use L'Hôpital's rule to check this.

Sol'n. $K = mc^2 - m_0c^2 = \dfrac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} - m_0 c^2$

$$= m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right)$$

v is always less than c, so we can write $(1-(\frac{v}{c})^2)^{-1/2}$ as a power series (binomial series) w/ $k = -\frac{1}{2}$ (we just did this).

$$\left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{-v^2}{c^2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{v^{2n}}{c^{2n}}$$

or $$= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots$$

where $x = \frac{-v^2}{c^2}$. So we have

$$K = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \right)$$

If v is much less than c, then $\frac{v^{2n}}{c^{2n}}$ is very small. If we omit all but the first term in the parentheses, we get

$$K \approx m_0 \cancel{c^2} \underbrace{\left(\tfrac{1}{2} v^2/\cancel{c^2}\right)}_{\text{this is } (T_1 - 1)} = \boxed{\frac{1}{2} m_0 v^2}$$

2. Use Taylor's formula to calculate the difference in these formulas when $|v| \leq 100$ m/s

Sol'n. Since we used T_1 to approximate $\left(1-\frac{v^2}{c^2}\right)^{-1/2}$, we want to find an upper bound on R_1.

(again $x = \frac{-v^2}{c^2}$)

Taylor's formula: $R_1(x) = \dfrac{f''(z)}{2!} x^2$