Ex. Find R and I of C:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

$$\lim \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right| = \frac{|x+2|}{3} \lim \left| \frac{n+1}{n} \right| = \frac{|x+2|}{3}$$

$$|x+2| < 3 \quad \Rightarrow \quad -5 < x < 1$$

$$\underline{R = 3}$$

Plug in -5: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n n$ diverges.

Plug in 1: $\displaystyle\sum_{n=0}^{\infty} \frac{n\, 3^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} n$ diverges

so, I of C: $\boxed{-5 < x < 1}$

10/22/12

8.6. Representing functions as power series

First, recall the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

That is, the function $f(x) = \frac{1}{1-x}$, $-1 < x < 1$, is equivalent to the power series above!

* Notice that we <u>must</u> specify the domain. These do not agree everywhere.

Ex. Similarly, represent $f(x) = \frac{1}{1+x^2}$ as the sum of a power series and find R, and IC (the domain).

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

this is a geometric series, so $R=1$.

$$-1 < x < 1$$

diverges at each end point.

Ex. $f(x) = \frac{1}{x+2}$ Same Question.

$$\frac{1}{2+x} = \frac{1}{2} \frac{1}{1+\frac{x}{2}} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

Converges when $\frac{|x|}{2} < 1$, or $|x| < 2$ so $R=2$.

Again, its geometric, so IC: $(-2,2)$.

Ex. $f(x) = \frac{x^3}{x+2}$

$$x^3 \cdot \frac{1}{x+2} = x^3 \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$$

This is still geometric, so $R=2$ and $IC = (-2,2)$.

Ex. 12. $f(x) = \dfrac{7x+1}{3x^2+2x-1}$

PFD: $3x^2+2x-1 = (3x-1)(x+1)$

$$\frac{7x+1}{3x^2+2x-1} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{5}{2}\frac{1}{3x-1} + \frac{3}{2}\frac{1}{x+1}$$

$$7x+1 = A(x+1) + B(3x-1)$$

$$-6 = B(-4) \Rightarrow B = \frac{3}{2}$$

$$\frac{7}{3}+1 = A\left(\frac{1}{3}+1\right) \Rightarrow \frac{10}{3} = \frac{4A}{3} \Rightarrow A = \frac{5}{2}$$

$$f(x) = \frac{-5}{2}\frac{1}{1-3x} + \frac{3}{2}\frac{1}{1+x}$$

$$= \boxed{\frac{-5}{2}\sum_{n=0}^{\infty}(3x)^n + \frac{3}{2}\sum_{n=0}^{\infty}(-x)^n}$$

$IC = \left(-\frac{1}{3}, \frac{1}{3}\right)$ b/c of the left half.

For 11. $f(x) = \dfrac{3}{x^2+x-2}$ also try completing the square.

$x^2+x-2 = \left(x+\frac{1}{2}\right)^2 - \frac{9}{4}$ so $f(x) = \dfrac{3}{\left(x+\frac{1}{2}\right)^2 - \frac{9}{4}}$

$$= -\frac{4}{9}\cdot\frac{3}{1-\frac{9}{4}\left(x+\frac{1}{2}\right)^2}$$

$$= -\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{9}{4}\left(x+\frac{1}{2}\right)^2\right]^n$$

compare w/ your solution using PFD. (i.e. check R and IC).

Differentiation and Integration of power series

If $\sum c_n (x-a)^n$ has radius of convergence $R>0$, then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is differentiable (hence also continuous) on the interval $(a-R, a+R)$, and

1.) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$$

2.) $$\int f(x)\,dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Equivalently,

1.) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$ and

2.) $$\int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx$$

That is, $\sum$ commutes w/ differentiation and integration.

Moreover, the radius of convergence is not changed (it's still R), <u>but</u> the end points may or may not converge any more.

Ex. The Bessel Function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n \, 2n \, x^{2n-1}}{2^{2n}(n!)^2}}$$

must remember to do this!

If you want to shift the index to start at 0 it will become:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cancel{2}(n+1) x^{2n+1}}{2^{2n+\cancel{2}1}((n+1)!)^2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1) x^{2n+1}}{2^{2n+1}((n+1)!)^2}}$$

Ex. Write $\frac{1}{(1-x)^2}$ as a power series.

Notice: $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right] = \sum_{n=1}^{\infty} n x^{n-1}$

or, $\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n}$

Ex. Write a power series representation for $f(x) = \ln(1-x)$.

$\int \frac{1}{1-x}\,dx = -\ln(1-x)$, so $\ln(1-x) = -\int \frac{1}{1-x}\,dx$

again $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $-\int \frac{1}{1-x}\,dx = -\sum_{n=0}^{\infty} \int x^n\,dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$

To determine the value of C, we put in $x=0$, and get

$0 + C = 0$, so $C = 0$

then $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Plug in $x = \frac{1}{2}$ to get

$$\ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

but $\ln\left(\frac{1}{2}\right) = -\ln(2)$, so

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n 2^n}.$$

Ex. Find a power series representation for $f(x) = \tan^{-1}(x)$

$$f(x) = \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Put in $x=0$ to get $C=0$, so

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Ex. 35. Show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a solution to the DE:

$$f'(x) = f(x)$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Ex. 37. Add this as an RE

Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ Find IC for f, f', f''.

For f: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \cdot \frac{n^2}{n^2+2n+1} \to |x| < 1$ so $R = 1$

$IC = [-1, 1]$

$$f'(x) = \sum_{n=2}^{\infty} \frac{n\, x^{n-1}}{n^2} = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{x^n}{n+1}$$ $R=1$ still

$IC = [-1, 1)$

$$f''(x) = \sum_{n=2}^{\infty} \frac{n\, x^{n-1}}{n-1} = \sum_{n=1}^{\infty} \frac{(n-1)\, x^n}{n-2}$$ $R=1$ still

$IC = (-1, 1)$

Ex. Integrate: $\int \frac{1}{1+x^7}\, dx$

$$\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

So $$\int \frac{1}{1+x^7}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{7n}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} + C$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + - \cdots$$

Converges for $|x| < 1$.

Ex. 35!. $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ Find R and I.C.

Show that $f'(x)=f(x)$.

Deduce that $f(x)=e^x$

Put $x=1$, and write out a series that adds up to e.

Ex. (21.) Find a P.S. rep. for $f(x)=\ln\left(\frac{1+x}{1-x}\right)$

Ex. (20.) $f(x)=\frac{1}{x^2+25}$.

Find P.S. rep. and graph a few S_n.

Ex. (32.) Show that $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ solves

$$f''(x)+f(x)=0.$$

Ex. (34.) $J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$

$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Show that $J_0'(x)=-J_1(x)$.

Ex. Use a Power series to approximate the definite integral to six decimal places

$$\int_0^{0.3}\frac{x^2}{1+x^4}\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+3}}{n+3}\Bigg|_0^{0.3}$$

$$\frac{x^2}{1+x^4}=\sum_{n=0}^{\infty}x^2(-x^4)^n=\sum_{n=0}^{\infty}(-1)^n x^{n+2}$$