8.5. Power Series

A <u>power series</u> is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and c_i are constants. (<u>coefficients</u>)

For any fixed x, this is a series of numbers that may or may not converge.

~~If the series converges, then~~ the power series is a function that we can think of as an "infinite polynomial."

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

<u>Ex</u>. $\sum_{n=0}^{\infty} x^n = f(x) = 1 + x + x^2 + x^3 + \cdots$

This series converges when $-1 < x < 1$ (geometric series).

so f is a function w/ domain $(-1,1)$

For any $a \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} (x-a)^n = 1 + (x-a) + (x-a)^2 + (x-a)^3 + \cdots$$

converges for $a-1 < x < a+1$.

This is called a power series <u>centered at a</u>.

(Really put in the c_n's)

Ex. For what values does $\sum_{n=0}^{\infty} n!\,x^n$ converge?

Sol'n. Use the ratio test.

$$a_{n+1} = (n+1)!\,x^{n+1} = (n+1)x\,n!\,x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\,x\,\cancel{n!}\,\cancel{x^n}}{\cancel{n!}\,\cancel{x^n}}\right| = |x|(n+1)$$

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim|x|(n+1) = \infty$$

So this series diverges for all $x \neq 0$, but converges for $x = 0$.

Ex. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ For what values of x does this converge?

$$a_{n+1} = \frac{(x-3)^n(x-3)}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(x-3)^n}(x-3)}{n+1}\cdot\frac{n}{\cancel{(x-3)^n}}\right| = \frac{n}{n+1}|(x-3)|$$

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{n}{n+1}|x-3| = |x-3|$$

By the Ratio Test, this series will converge if the limit is < 1, or $|x-3| < 1$

$$\Rightarrow\ -1 < x-3 < 1$$
$$2 < x < 4$$

We need to plug in the end points individually:

at $x=2$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by AST.

$x=4$: $\sum_{n=1}^{\infty}\frac{1^n}{n} = \sum\frac{1}{n}$ diverges by p-test.

So Radius of Conv. is $\boxed{2 \le x < 4}$

<u>Ex</u>. The 0^{th} Bessel function:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Find its domain.

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2} ((n+1)!)^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right|$$

$$= \lim \left| \frac{x^2}{4(n+1)^2} \right| = 0$$

so $\sum$ converges for all x, and $D(J_0) = \mathbb{R}$!

Graph some Bessel partial sums?

<u>Ex</u>. Find the radius of convergence R and the interval of convergence of:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$\lim \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \lim \left| \frac{-3\sqrt{n+1}}{\sqrt{n+2}} \right| |x| = 3|x|$$

need: $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$, so $R = \frac{1}{3}$

$$-\frac{1}{3} < x < \frac{1}{3}$$

By p-test and AST: $\boxed{-\frac{1}{3} < x \leq \frac{1}{3}}$

__Ex__. Find R and I of C:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

$$\lim \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right| = \frac{|x+2|}{3} \lim \left| \frac{n+1}{n} \right| = \frac{|x+2|}{3}$$

$$|x+2| < 3 \quad \Rightarrow \quad -5 < x < 1$$

$$\underline{R = 3}$$

Plug in -5: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n n$ diverges.

Plug in 1: $\displaystyle\sum_{n=0}^{\infty} \frac{n\,3^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} n$ diverges

so, I of C: $\boxed{-5 < x < 1}$

10/22/12

__8.6__. __Representing functions as power series__

First, recall the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

That is, the function $f(x) = \frac{1}{1-x}$, $-1 < x < 1$, is equivalent to the power series above!

* Notice that we __must__ specify the domain. These do not agree everywhere.