8.4. Other Convergence Tests

The <u>Alternating Series Test</u>:

If the alt. series $\sum (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + - \cdots$ $\quad (b_n > 0)$

satisfies

$\quad$ i.) $b_{n+1} \leq b_n$ for all n

and ii.) $\lim b_n = 0$

then the series is convergent.

Figure 2. on pg. 439 illustrates why this works for the alternating harmonic series.

<u>Ex</u>. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ so $b_n = \frac{1}{n}$

$$b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n} \quad \text{for all } n \geq 1.$$

also $\lim \frac{1}{n} = 0$, therefore the series converges by AST.

<u>Ex</u>. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$, $b_n = \frac{3n}{4n-1}$, $\lim b_n = \frac{3}{4} \neq 0$, so diverge. by TFD.

<u>Ex</u>. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$, $b_n = \frac{n^2}{n^3+1}$, $\lim b_n = 0$, but is $\frac{n^2}{n^3+1}$

decreasing? $\displaystyle (b_n)' = \frac{(n^3+1)(2n) - (n^2)(3n^2)}{(n^3+1)^2} = \frac{2n^4 + 2n - 3n^4}{(n^3+1)^2}$

$$= \frac{2n - n^4}{(n^3+1)^2} \quad \text{for } n \geq 2 \text{ this is decreasing. (neg.)}$$

so $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is conv. by AST.

A.S. Estimation Thm.

If $\sum (-1)^n b_n$ is a conv. alt. series w/ $0 \le b_{n+1} \le b_n$ and $\lim b_n = 0$, then

$$|R_n| = |s - s_n| \le b_{n+1}$$

RE: Proof. s lies between any two partial sums, so

$$|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}. \qquad \square$$

Ex. Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to 3 decimal places.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + - \cdots$$

$$= \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + - \cdots$$

$$\frac{1}{5040} < \frac{1}{5000} = 0.0002$$

So, $|s - s_6| \le b_7 < 0.0002$

Add up the first 7: $s \approx 0.368$

Defn. A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent.

Of course $-\sum |a_n| \le \sum a_n \le \sum |a_n|$, so $\sum a_n$ is also convergent in this case.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely conv. by p-test.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conv by AST, but is not abs. conv. (p-test).

Such a series is called conditionally convergent.

Ex. C or D: $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ $\quad a_n = \frac{\cos n}{n^2}, \ |a_n| = \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-test, so $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ conv. by CT.

$\therefore \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

The Ratio Test :

1. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then $\sum a_n$ is divergent.
   (or $= \infty$)
3. If $\lim \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Rat. test is inconclusive.

Ex. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ Test for abs. conv.

$a_n = \frac{(-1)^n n^3}{3^n}$ $\quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cancel{(-1)^{n+1}} (n+1)^3}{\cancel{3^{n+1}}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} n^3} \right|$

$a_{n+1} = \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}}$ $\quad = \left| \frac{n^3 + 3n^2 + 3n + 1}{3n^3} \right|$

$\lim \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} < 1$, so abs. conv!

Ex. C or D: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n^n}{n!} \qquad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \left(1 + \frac{1}{n}\right)^n = e > 1$$

10/9/12 So the series diverges by the rat. test!

The Root Test:

1. If $\lim \sqrt[n]{a_n} = L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\lim \sqrt[n]{a_n} = L > 1$ (or $= \infty$), then $\sum a_n$ is divergent.
3. If $\lim \sqrt[n]{a_n} = 1$, then the root test is inconclusive.

Ex. Test $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ for conv.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$$

$$\lim \sqrt[n]{a_n} = \lim \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

So convergent!

✓ Some REs:

18. For what p is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ convergent? [All $p > 0$.]

42.* Srinivasa Ramanujan: (~1910)

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26390n)}{(n!)^4 \; 396^{4n}} = \frac{1}{\pi} \quad !$$

1985: William Gosper used this to calculate the first 17 million digits of π!