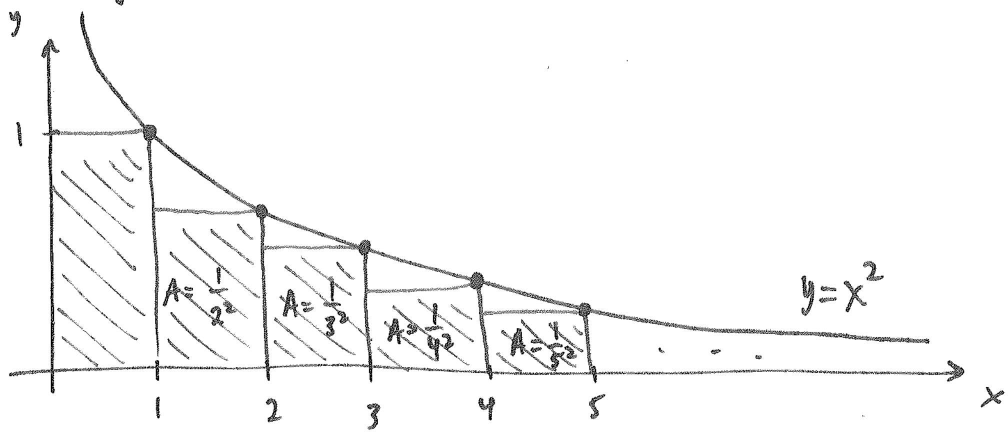
8.3. Integral and Comparison Tests

Ex. We want to know whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.



The area of the first box is 1, which is obviously finite. If we ignore it, the rest of the boxes lie below the curve, so their areas add up to less than the integral of the function:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

So, if $\int_1^{\infty} \frac{1}{x^2}dx < \infty$, then so is $\sum_{n=1}^{\infty} \frac{1}{n^2}$!

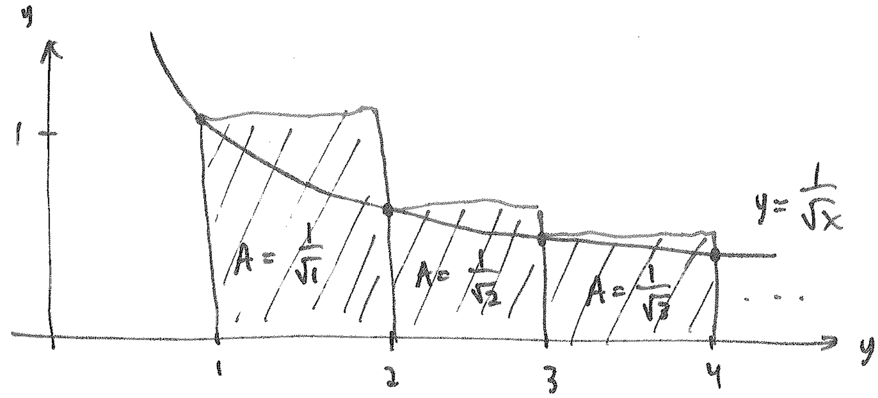
We know $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

We still don't know its exact sum, but this is a good start.

$\frac{\pi^2}{6}$! :)

Ex. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Use the same idea (kind of).



$$\text{So } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$$

We know that $\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ diverges, so the series must diverges as well.

This leads us to the Integral Test:

Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ such that $a_n = f(n)$ for $n \in \mathbb{N}$. Then $\sum(a_n)$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent. i.e.,

a. If $\int_1^{\infty} f(x)\,dx < \infty$, so is $\sum a_n$.

b. If $\int_1^{\infty} f(x)\,dx$ is divergent, so is $\sum a_n$.

* Note: We don't actually need to start at 1. Since partial sums are always finite, we only need to check a "tail" of the series. We can choose any lower limit that is "convenient."

Ex. Conv. or div.? : $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ ← decreasing? check via $\frac{d}{dx}$.

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} (\ln t)^2 = \infty,$$

So the series diverges.

Remember the p-test for integrals?

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$, diverges otherwise.

This gives us a p-test for series:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges otherwise. (≤ 1)

Testing by Comparison.

Consider $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. Convergent or divergent?

$0 < \frac{1}{2^n+1} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$

We know $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, so $\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

This series is convergent.

We have the Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series w/ positive terms, and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

a) If $\sum b_n$ is convergent, then so is $\sum a_n$

b) If $\sum a_n$ is divergent, then so is $\sum b_n$.

Ex. C or D: $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} =: \sum a_n$

$$0 < \frac{5}{2n^2+4n+3} < \frac{5}{2}\cdot\frac{1}{n^2} \quad \forall n \in \mathbb{N}$$

and $\displaystyle\frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so $\sum a_n$ also converges!

Ex. back to $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$. We just showed that this div. w/ integral test. Also can use comp. test:

$$\frac{1}{n} < \frac{\ln n}{n} \quad \forall n \geq 3,$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so $\sum\frac{\ln n}{n}$ must also.

Ex. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n-1}$

try to compare to $\sum\frac{1}{2^n}$, which is convergent, but

$$\frac{1}{2^n-1} > \frac{1}{2^n} \quad \forall n \in \mathbb{N},$$

so the comparison test tells us nothing useful.
Instead,

The Limit Comparison Test:

Suppose $\sum a_n$, $\sum b_n$ are series w/ positive terms. If

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0 \quad (\text{in particular, } \neq 0)$$

then either both series converge, or both diverge.

Proof. Let $m, M > 0$ such that $m < c < M$. Since $\lim \frac{a_n}{b_n} = c$, then $\exists\, N \in \mathbb{N}$ s.t.

$$m < \frac{a_n}{b_n} < M \qquad \forall\, n > N$$

$$\Rightarrow \quad m b_n < a_n < M b_n$$

If $\sum b_n$ converges, so does $M \sum b_n$, thus $\sum a_n$ converges by the comparison test. If $\sum b_n$ diverges, then so does $m \sum b_n$, and $\sum a_n$ diverges by the comparison test. $\square$

Ex. Use LCT to show $\sum \frac{1}{2^n - 1}$ converges.

Compare to $\sum \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{x\to\infty} \frac{2^x}{2^x - 1} \overset{L'H}{=} \lim_{x\to\infty} \frac{\ln(2)\, 2^x}{\ln(2)\, 2^x} = 1 > 0$$

$\therefore \sum \frac{1}{2^n - 1}$ converges.

Moct

RE 25. C or D: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Compare to $\sum \frac{1}{n}$ via LCT

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \overbrace{\lim_{u\to 0} \frac{\sin u}{u}}^{\text{F.T.I.S.}} = 1.$$

$\sum \frac{1}{n}$ diverges, so $\sum \sin\left(\frac{1}{n}\right)$ also diverges!