8.2. Series

Let $(a_n) = (a_n)_{n=1}^{\infty}$ be a sequence. If we add up all of the terms we get

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots =: \sum_{n=1}^{\infty} a_n = \sum a_n$$

Does it make sense to add up infinitely many things?

This is called an <u>infinite series</u>.

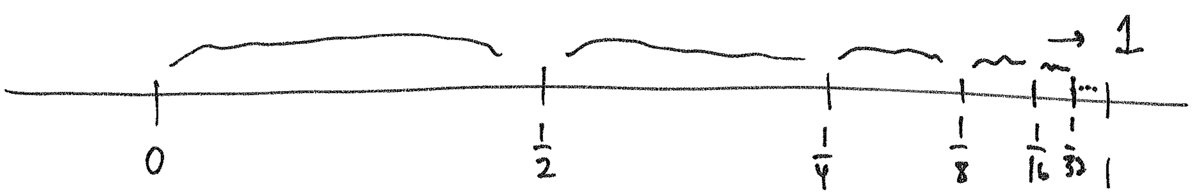
<u>Ex</u>. $1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$

We get ∞ (not a number), so we say that the series $\sum_{n=1}^{\infty} n$ diverges.

<u>Ex</u>. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$

We add smaller and smaller terms. This is a <u>geometric series</u> (from HS algebra). We know that the sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}(2) = 1.$$



<u>Idea</u>:

Let $s_1 = a_1$

$s_2 = a_1 + a_2 = s_1 + a_2$

$s_3 = a_1 + a_2 + a_3 = s_2 + a_3$

$\vdots$

$s_n = a_1 + \cdots + a_n = s_{n-1} + a_n$

These get closer and closer to 1.

We write $s_n = \sum_{i=1}^{n} a_i$ and call this the <u>n^{th} partial sum</u> of the series $\sum a_n$.

Notice that $s_\infty = \sum_{i=1}^{\infty} a_i = s$ is the sum of the whole series, if it exists.

We write:

$$\boxed{s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i}$$

If this limit exists, s is called the <u>sum</u> of the series. If not, we say the series diverges.

<u>Prop.</u> The geometric series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

is convergent if $|r| < 1$, and its sum is

$$s = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, then the geometric series diverges.

<u>Proof.</u> Let $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$

then $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

and shift.

$s_n - rs_n = a - ar^n$

or $\boxed{s_n = \dfrac{a(1-r^n)}{1-r}}$

So, the n^{th} partial sum is given by

$$s_n = \frac{a(1-r^n)}{1-r}$$

then $$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a\left(1-\lim_{n\to\infty} r^n\right)}{1-r}$$

$$= \frac{a}{1-r} \quad \text{if } |r|<1 \quad \text{and}$$

$$\text{diverges if } |r| \geq 1$$

If $r=1$ or $r=-1$, then it's easy to see that

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges.} \qquad \square$$

<u>Ex</u>. Find the sum

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + - \cdots$$

$$= \sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^{n-1} = \frac{5}{1+\frac{2}{3}} = \frac{5}{5/3} = \boxed{3}$$

<u>Ex</u>. $\displaystyle\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ conv. or div.?

$$= \sum_{n=1}^{\infty} 4^n \cdot 3 \cdot 3^{-n}$$

$$= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{4}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1} \qquad r = \frac{4}{3} > 1, \text{ so } \underline{\text{div.}}$$

Ex. Write the number $2.3\overline{17} = 2.3171717717...$ as a decimal.

$$2.3\overline{17} = 2.3 + .017 + .00017 + .0000017 + \cdots$$

$$= 2.3 + 17(10^{-3}) + 17(10^{-5}) + 17(10^{-7}) + \cdots$$

$$= 2.3 + \frac{17}{10^3} + \frac{17}{10^3}\cdot\frac{1}{10^2} + \frac{17}{10^3}\cdot\frac{1}{10^4} + \cdots$$

$$= 2.3 + \sum_{n=1}^{\infty} \frac{17}{10^3}\left(\frac{1}{10^2}\right)^{n-1}$$

$$= \frac{230}{100} + \frac{17}{1000}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= \frac{230}{100} + \frac{17\cdot \cancel{100}}{99\cdot \cancel{1000}} = \frac{23}{10} + \frac{17}{990} = \frac{99\cdot 23 + 17}{990}$$

$$= \frac{2294}{990} = \boxed{\frac{1147}{495}}$$

Ex. Find the sum of the series $\sum_{n=0}^{\infty} x^n$, $|x|<1$.

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

This is a function of x w/ domain $(-1,1)$!

Ex. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{n(n+1)}$$

Ew.

Insted: PFD $\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1} = \frac{1}{i} - \frac{1}{i+1}$

Add this insted:

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} + - \cdots + \cancel{\tfrac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

"telescoping" sum.

And $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$

Ex. Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(p.424)

RE

get, in general, $S_{2^n} > 1 + \frac{n}{2}$

$\lim S_n = \lim S_{2^n}$ diverges!

RE: Theorem. $\sum_{n=1}^{\infty} a_n$ is convergent implies $\lim_{n\to\infty} a_n = 0$.

Proof. $S_n = a_1 + \dots + a_n$

$S_{n+1} = a_1 + \dots + a_{n+1}$

$\Rightarrow a_{n+1} = S_{n+1} - S_n$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - S_n = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = S - S = 0.$ $\square$

Test for divergence: If $\lim_{n\to\infty} a_n$ = dne or $\neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex. Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$\lim a_n = \frac{1}{5} \neq 0.$

Thm. Spse $\sum a_n$ and $\sum b_n$ are conv. series. Then so are the series:

i) $\sum c a_n = c \sum a_n$ $\qquad c \in \mathbb{R}$.

ii) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

Ex. Find the sum of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$

$$= 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3(1) + 1 = \boxed{4}$$
