Chapter 8: Series

8.1 : Sequences

A sequence is a function $a(n) = a_n$ whose domain is

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

or

A sequence is an ordered list of numbers

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

we write either (a_n) or $(a_n)_{n=1}^{\infty}$

Ex. $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+1}$ $\quad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$

$\left(\sqrt{n-3}\right)_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3},\ n \geq 3$ $\quad \{0, 1, \sqrt{2}, \sqrt{3}, 2, \dots, \sqrt{n-3}, \dots\}$

this gives the same sequence as

$$a_n = \sqrt{n},\ n \geq 0$$

Ex. Find a formula for the general term

a.) $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\}$ $\quad a_n = (-1)^n \frac{n}{(n+1)^2}$

b.) $\{2, 7, 12, 17, \dots\}$ $\quad a_n = 2 + 5(n-1) = -3 + 5n$

c.) $\{5, 1, 5, 1, 5, 1, \dots\}$ $\quad a_n = \begin{cases} 5 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$

Some sequences don't have nice formulas.

Ex. The Fibonacci Sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$$

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_{n+2} &= a_{n+1} + a_n \end{aligned}\right\}$$ recursive definition: the next one depends on the previous.

Ex. $a_n = n^{th}$ decimal place of the natural number

$$\{7, 1, 8, 2, 8, 1, 8, 2\,8, 4, 5, \ldots\}$$

Never "repeats" because e is irrational.

We want to do calculus on/with sequences!

Defn. A sequence (a_n) has the limit L and we write

$$\lim_{n\to\infty} a_n = \lim a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if and only if for every $\varepsilon > 0$ there exists an integer N such that if $n > N$, then $|a_n - L| < \varepsilon$.

i.e., we can make the terms a_n as close to L as we want by taking a large enough n.

If (a_n) has a limit we say the sequence converges. Otherwise, it diverges.

Ex. Prove that $a_n = \frac{n}{2n+1}$ has limit $\lim a_n = \frac{1}{2}$.

Fix $\varepsilon > 0$, if $n > N =$ ____ then

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n - 2n - 1}{4n+2}\right| = \left|\frac{1}{2(2n+1)}\right| < \varepsilon$$

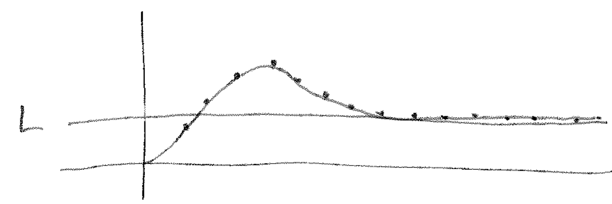
$$n > \frac{1}{4\varepsilon} + \frac{1}{2} = \frac{2\varepsilon + 1}{4\varepsilon}$$

$$2n+1 > \frac{1}{2\varepsilon}$$

$$2n > \frac{1}{2\varepsilon} + 1$$

$>$

Thm. If f is a real function ~~and~~ s.t. $f(n) = a_n$, and $\lim_{x \to \infty} f(x) = L$, then $\lim a_n = L$.



In particular, $\lim_{n \to \infty} \frac{1}{n^r} = 0$ if $r > 0$.

Defn. $\lim a_n = \infty$ means that for every positive number M, there is an integer N such that if

$$n > N, \text{ then } a_n > M$$

Such a limit diverges "to infinity".

Limit rules: If (a_n) and (b_n) are convergent sequences and $c \in \mathbb{R}$, then

1. $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$
2. $\lim (c\, a_n) = c \lim a_n$
3. $\lim (a_n \cdot b_n) = \lim a_n \cdot \lim b_n$
4. $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if $\lim b_n \neq 0$.
5. $\lim a_n^p = [\lim a_n]^p$ if $p > 0$ and $a_n > 0$

Squeeze Thm. If $a_n \le b_n \le c_n$ for $n \ge N$, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Ex. Calculate $\lim \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$

Ex. Convergent or divergent?

$$(a_n) = \{-1, 1, -1, 1, -1, 1, -1, 1, \dots\}$$

D.

~~Ex.~~ Theorem. If $\lim |a_n| = 0$, then $\lim a_n = 0$.

Ex. Evaluate $\lim \frac{(-1)^n}{n}$

$$\lim\left|\frac{(-1)^n}{n}\right| = \lim \frac{1}{n} = 0, \text{ so } \lim \frac{(-1)^n}{n} = 0 \text{ also.}$$

Ex. $a_n = \frac{n!}{n^n}$ Converge or diverge?

$n! = 1\cdot 2\cdot 3\cdot \cdots n$

$n^n = n\cdot n\cdot n \cdots n$

so $a_n = \frac{1}{n}\left(\frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n}\right)$ looks like these are decreasing to 0?

this ≤ 1

so $0 < a_n \le \frac{1}{n}$

Now, use squeeze thm! get $\lim a_n = 0$. ☺

Ex. $\lim r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \end{cases}$ diverges if $r > 1$.

Defn. A sequence is called increasing if $a_{n+1} > a_n$ for all n. decreasing if $a_{n+1} < a_n$ for all n. Monotonic if it's either inc or dec.

Ex. Show that $a_n = \frac{3}{n+5}$ is decreasing

$$a_{n+1} = \frac{3}{n+1+5} = \frac{3}{n+6} < \frac{3}{n+5} = a_n \quad \text{for all } n.$$

Ex. $a_n = \frac{n}{n^2+1}$ $\quad a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+1+1}$ dec.

Cross multiply to get

$$n^3+2n^2+2n \overset{?}{>} (n+1)(n^2+1) = n^3+n^2+n+1$$

$$n^2+n > 1 \quad \checkmark \qquad \text{since } n \geq 1.$$

Defn. A sequence is bounded above if $\exists\, M$ s.t.

$$a_n \leq M \quad \text{for all } n.$$

bounded below if $\exists\, m$ s.t.

$$m \leq a_n \quad \text{for all } n.$$

If $m \leq a_n \leq M$, then a_n is bounded.

Theorem. Monotone Sequence Thm. Every bounded, monotonic sequence is convergent.

Proof. In the book. Justify w/ a picture.

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limits:

Ex. $a_n = \left(1+\frac{2}{n}\right)^n$

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