7.6. Differential Equations

A differential equation is an equation involving an unknown function and one or more of its derivatives.

Ex. $y' = xy$ We want to solve for y.

$$\frac{dy}{dx} = xy$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$e^{\ln y} = e^{\frac{1}{2}x^2 + C}$$

$$y = e^{\frac{1}{2}x^2 + C} = e^{C} e^{\frac{1}{2}x^2} = K e^{\frac{1}{2}x^2}$$ where k is an unknown constant.

An equation like this is called separable, because we can separate x and y.

In general, separable equations look like:

$$\frac{dy}{dx} = f(x)\,g(y)$$

so, we can solve $\frac{dy}{g(y)} = f(x)\,dx$

A better (or easier) situation:

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$ so that

$$h(y)\,dy = f(x)\,dx.$$

Ex. $\frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow$

$$y^2\,dy = x^2\,dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$y^3 = x^3 + C$$

$$\boxed{y = \sqrt[3]{x^3 + C}}$$

If we want to know C, then we need an <u>initial condition</u>.

Suppose ~~y(0)~~ we want (or know) that $y(0) = 2$. Then

$$y(0) = \sqrt[3]{0^3 + C} = \sqrt[3]{C} = 2 \Rightarrow C = 2^3 = 8$$

and $y = \sqrt[3]{x^3 + 8}$ is the <u>particular solution</u> to this problem.

<u>Ex</u>. The DE $y' = ky$ occurs in nature all of the time. It says that the growth (y') of a quantity (y) is proportional to the ~~[illegible]~~ quantity itself. e.g. pop'n growth, interest, radioactive decay

Let's solve it. Assume $y = y(t)$.

$$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt$$

$$\Rightarrow e^{\ln y} = e^{kt + C}$$

$$y = e^C e^{kt} = Ce^{kt}$$

$$\boxed{y = Ce^{kt}}$$

Eg. $\begin{cases} \text{Pop'n:} & P = P_0 e^{kt} \\ \text{Int:} & A = Pe^{rt} \end{cases}$

Recall that $\frac{dy}{dx}$ is the slope of the tangent line of a function. So $\frac{dy}{dx} = F(x,y)$ tells us what the slope ~~of a function~~ ~~y~~ looks like at a point (x,y).

We can graph these slopes as small line segments. Then a solution to a DE is obtained by choosing a starting point and following the slopes to draw a graph. These graphs are called slope fields, direction fields, flow fields, etc.

Ex. (10) $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$ $\Rightarrow$ $\displaystyle\int \frac{1+y^2}{y}\,dy = \int \cos x\,dx$

$y(0) = 1$

$$\ln y + \tfrac{1}{2}y^2 = \sin x + C$$

$$\cancel{\ln 1} + \tfrac{1}{2}1^2 = \cancel{\sin 0} + C$$

$$C = \tfrac{1}{2}$$

So, soln is $\boxed{\ln y + \tfrac{1}{2}y^2 = \sin x + \tfrac{1}{2}}$

Can't simplify this much more.

Ex. (7) $\dfrac{du}{dt} = 2 + 2u + t + tu$

$= \cancel{\cos}\,2(1+u) + t(1+u)$

$= (2+t)(1+u)$

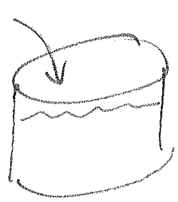
$\Rightarrow$ $\dfrac{du}{1+u} = (2+t)\,dt$

$$\ln|1+u| = 2t + \tfrac{1}{2}t^2 + C$$

$$1+u = Ce^{2t+\frac{1}{2}t^2}$$

$$\boxed{u(t) = Ce^{2t+\frac{1}{2}t^2} - 1}$$

Mixing Problems



5000 L of water
w/ 20 kg of dissolved salt

In: water enters tank at 25 L/min
w/ 0.03 kg/L of salt dissolved

Out: Thoroughly mixed solution is drained at same rate.

Q: Find how much salt is in tank after half an hour.

~~Instead of $x=y^2$, find arc length of $y=2\sqrt{x}$ from 0 to 1~~

~~doesn't look good.~~

Let $y(t)$ = amt of salt at time t

We know $y(0)=20$. We want $y(30)$.

We know $\frac{dy}{dt}$ = (rate in) − (rate out) in kg/min

<u>rate in:</u> $(0.03 \text{ kg/L})(25 \text{ L/min}) = \frac{3 \cdot 25}{100} \text{ kg/min} = .75 \text{ kg/min}$

<u>rate out:</u> Conc. at time t is $\frac{y(t) \text{ kg}}{5000 \text{ L}}$ since tank always has 5000 L.

so, $\left(\frac{y}{5000} \frac{\text{kg}}{\text{L}}\right)\left(25 \text{ L/min}\right) = \frac{y}{200}$ kg/min

Thus, our DE is

$$\frac{dy}{dt} = .75 - \frac{y}{200} = \frac{150-y}{200}$$

$$\boxed{y(30) \approx 38.1 \text{ kg}}$$

$$\frac{dy}{150-y} = \frac{1}{200}\, dt$$

$$-\ln(150-y) = \frac{t}{200} + C$$

$y(0)=20$ gives $C = -\ln(130)$

So $-\ln(150-y) = \frac{t}{200} - \ln(130)$

$$\frac{1}{150-y} = \frac{1}{130} e^{t/200}$$

$$150 - y = \frac{130}{e^{t/200}}$$

$$\boxed{y = 150 - 130 e^{-t/200}}$$

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