7.5. Applications to Physics

Work:

Newton's 2^{nd} Law of motion: $F = ma$ or $F = m\frac{d^2s}{dt^2}$

where $s(t)$ is the position function of a particle.

Work = Force x distance

$$W = Fd$$

This works great as long as the force is constant. But if force is a function, then

$$\boxed{W = \int_a^b f(x)\,dx}$$

is the work done by f between $x=a$ and $x=b$.

Ex. When a particle is located x units from the origin, a force of x^2+2x pounds acts on it. How much work is done in moving it from $x=1$ to $x=3$?

$$W = \int_1^3 x^2 + 2x\,dx = \frac{1}{3}x^3 + x^2\Big|_1^3 = \frac{27}{3} + 9 - \frac{1}{3} - 1$$

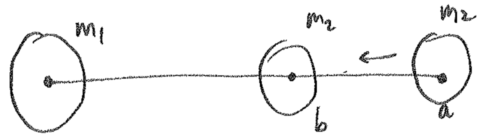
$$= 27 - \frac{4}{3} = \frac{54-4}{3} = \boxed{\frac{50}{3}} \text{ ft-lb.}$$

Ex. (21) Newton's law of gravitation: two bodies w/ masses m_1 and m_2 attract each other w/ a force

$$F = G\,\frac{m_1 m_2}{r^2}$$

where r is the distance between their ~~centers~~ centers and G is the gravitational constant.

If one body is fixed, find the work done to move the other from $r=a$ to $r=b$



$$W = \int_a^b G\,\frac{m_1 m_2}{r^2}\,dr = G m_1 m_2 \int_a^b r^{-2}\,dr$$

$$= G m_1 m_2 \left(-\frac{1}{r}\right)\Big|_b^a = \left| G m_1 m_2 \frac{1}{b} - G m_1 m_2 \frac{1}{a} \right|$$

Compute the work required to launch a 1000 kg satellite vertically to an orbit 1000 km high.

Take $m_E = 5.98 \times 10^{24}$ kg

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$ and

$r_E = 6.37 \times 10^6$ m

so $b = 6.37 \times 10^6$ and $a = 6.37 \times 10^6 + 10^6 = 7.37 \times 10^6$

Plug and chug to get:

$$\approx \boxed{8.496 \times 10^9 \text{ Joules } !}$$

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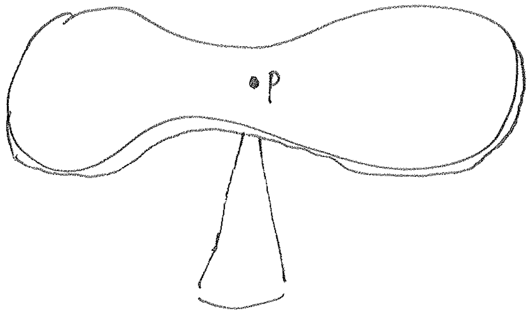
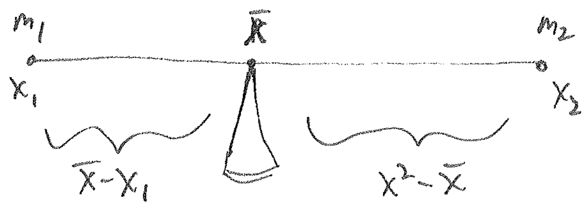
Moments and Centers of Mass:



P = the center of mass (gravity)

If this shape were a thin plate, it would balance perfectly at this point.

First, look at this problem as a 1-d problem:



Two masses are attached to opposite ends of a rod. We do we need to put the fulcrum so that the rod will balance?

Archimedes discovered the "Law of the lever":

$$m_1 d_1 = m_2 d_2$$

$$m_1(\bar{x}-x_1) = m_2(x_2-\bar{x}) \qquad \text{solve for } \bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

In general, if a system has n particles:

$$\boxed{\bar{x} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i}, \quad \text{where } m = \sum_{i=1}^{n} m_i$$

The quantity $\boxed{M := \sum_{i=1}^{n} m_i x_i}$ is called the <u>moment</u> of the system about the origin.

In a 2D system w/ particles at $(x_1,y_1) \cdots (x_n,y_n)$, there are moments

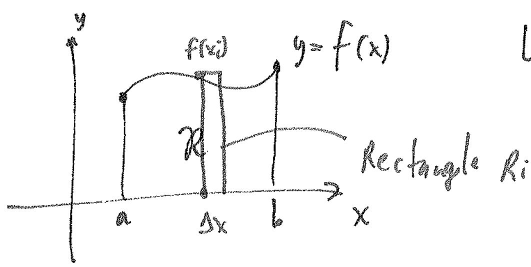
$$\begin{cases} M_y = \sum_{i=1}^{n} m_i x_i & \text{and} \\ M_x = \sum_{i=1}^{n} m_i y_i \end{cases}$$

M_y measures tendency to rotate about y-axis and M_x around x-axis.

The center of mass of a 2D system is given by

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \qquad \text{where } m = \sum m_i .$$

As usual, when we switch from discrete to continuous data, the sums become an integrals.



Let $\mathcal{R}$ be a flat plate w/ uniform density ρ.

For any rectangle, Area $= f(x_i)\Delta x$ so its mass is $\rho f(x_i)\Delta x$

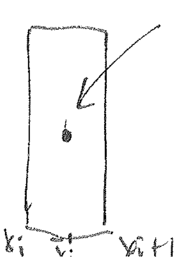
The moment of R_i about y-axis is $M_y(R_i) = [\rho f(x_i)\Delta x]\underbrace{\bar{x}_i}_{\text{distance to y-axis}}$.

As an end result:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(x_i)\Delta x = \boxed{\rho \int_a^b x f(x)\,dx .}$$

For M_x

$C = (\bar{x}_i, \frac{1}{2} f(x_i))$



$$M_x(R_i) = [\rho f(\bar{x}_i)\Delta x] \underbrace{\tfrac{1}{2} f(x_i)}_{\text{distance to x-axis.}}$$

Then $M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \boxed{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}$

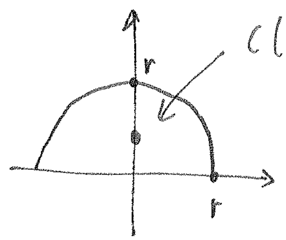
As before:

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} = \boxed{\frac{1}{A}\int_a^b x f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho \int_a^b f(x)\,dx} = \boxed{\frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx}$$

* center of mass is independent of the density (as long as density is constant).

Ex. Find the center of mass of A semi-circular plate.



$C(\bar{x},\bar{y})$ is somewhere around here.

$f(x) = \sqrt{r^2 - x^2}$ $\quad a = -r,\ b = r$

$A = \frac{1}{2}\pi r^2$ by geometry.

$u(r) = 0$?!
$u(-r) = 0$?!

$u = r^2 - x^2$
$du = -2x\,dx$

So, $\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx = \frac{2}{\pi r^2}\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx$

since u is even!

$$= -\frac{2}{\pi r^2}\int_0^0 \sqrt{u}\,du = \frac{-2}{\pi r^2}\left[\frac{2}{3}u^{3/2}\right]_0^0 = 0$$

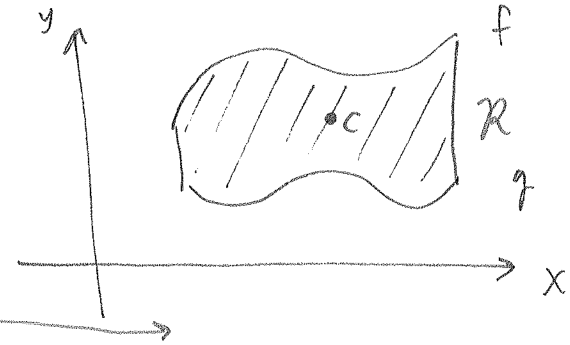
OR, by symmetry principle: center of mass must lie on axis of symmetry.

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\pi r^2}\int_{-r}^{r} r^2 - x^2\,dx = \frac{2}{\pi r^2}\int_0^r r^2 - x^2\,dx$$

$$= \frac{2}{\pi r^2}\left[r^2 x - \frac{1}{3}x^3\right]_0^r = \frac{2}{\pi r^2}\left[r^3 - \frac{1}{3}r^3\right] = \frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \boxed{\frac{4r}{3\pi}}$$

So, center of mass is at $\boxed{(\bar{x},\bar{y}) = \left(0, \frac{4r}{3\pi}\right)}$

What about a region like this:

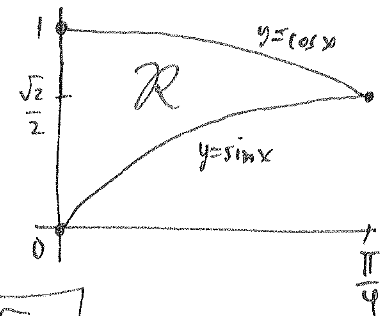


"centroid" of $\mathcal{R}$ is given by

$$\begin{cases} \bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx \\ \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx \end{cases}$$

Ex in text:
$y = x$, $y = x^2$
$x = 0$, $x = 1$
Ans: $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$

Ex (41) $y = \sin x$ $y = \cos x$ $x = 0$, $x = \frac{\pi}{4}$



$$A = \int_0^{\pi/4} \cos x - \sin x \, dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} = \frac{2\sqrt{2}}{2} - 1 = \boxed{\sqrt{2} - 1}$$

$$\bar{x} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\cos x - x\sin x \, dx = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\cos x \, dx - \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} x\sin x \, dx$$

$$= \frac{1}{\sqrt{2}-1}\left[x\sin x + \cos x - \left(-x\cos x + \sin x\right)\right]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}-1}\left[x\sin x + x\cos x + \cos x - \sin x\right]_0^{\pi/4} = \frac{\sqrt{2}\,\pi - 4}{4(\sqrt{2}-1)} \approx 0.267$$

$$\bar{y} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \frac{1}{2}(\cos x - \sin x)^2 \, dx = \frac{1}{2(\sqrt{2}-1)}\int_0^{\frac{\pi}{4}} \cos^2 x - 2\sin x\cos x + \sin^2 x \, dx$$

$$= \frac{1}{2(\sqrt{2}-1)}\int_0^{\frac{\pi}{4}} 1 - 2\sin x\cos x \, dx = \frac{1}{2(\sqrt{2}-1)}\left[x + \cos^2 x\right]_0^{\pi/4}$$

$$= \frac{1}{2(\sqrt{2}-1)}\left[\frac{\pi}{4} + \frac{1}{2} - 1\right] = \frac{\pi - 2}{8(\sqrt{2}-1)} \approx 0.345$$

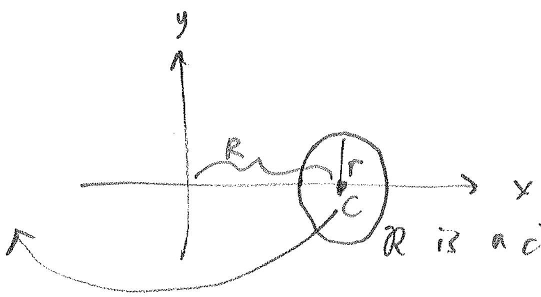
So, $\boxed{C = (\bar{x}, \bar{y}) = \left(\frac{\sqrt{2}\,\pi - 4}{4(\sqrt{2}-1)}, \frac{\pi - 2}{8(\sqrt{2}-1)}\right) \approx (0.267, 0.345)}$

This leads to a remarkable theorem:

<u>The Theorem of</u> $\boxed{\text{Pappus}}$: ← Greek mathematician, lived in 300s AD (CE).

Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the same plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$ (i.e., the circumfer of the circle created by rotating the centroid).

We've already used this theorem! The torus:



$\mathcal{R}$ is a circle of radius r

Volume = "$A(\mathcal{R})$" × "$d(C)$"

$A(\mathcal{R}) = \pi r^2$

$d(C) = 2\pi R$

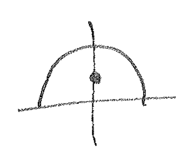
So, $\boxed{V = 2\pi^2 r^2 R}$

Proof of this is in the book, but we already justified it using Cavalieri's principle (loosely).

(48)

<u>Ex</u>. Use Pappus' Theorem to find the volume of a sphere of radius r.

We've already found that $C = \left(0, \frac{4r}{3\pi}\right)$ for



And area of semi-circle is $\frac{1}{2}\pi r^2$. So

$$V = \frac{1}{\cancel{2}}\pi r^2\left(\cancel{2}\cancel{\pi}\cdot\frac{4r}{3\cancel{\pi}}\right) = \boxed{\frac{4}{3}\pi r^3} \quad !$$

(49) is a "good problem".