$$\frac{1}{2}\int_0^{\operatorname{atan}(2)} \sec^3\theta \, d\theta$$

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$$\int \sec^3\theta \, d\theta$$

$u = \sec\theta \qquad dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta \qquad v = \tan\theta$

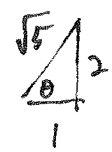
$$= \sec\theta\tan\theta - \int \sec\theta \underbrace{\tan^2\theta}_{\sec^2\theta - 1} \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$\Rightarrow \quad 2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$\theta = \arctan(2)$ means $\tan\theta = \frac{2}{1}$



$\tan 0 = 0$

$\sec 0 = 1$

$\sec\theta = \sqrt{5}$

So, $$L = \frac{1}{2}\int_0^{\arctan(2)} \sec^3\theta \, d\theta = \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| \Big|_0^{\operatorname{atan}(2)}$$

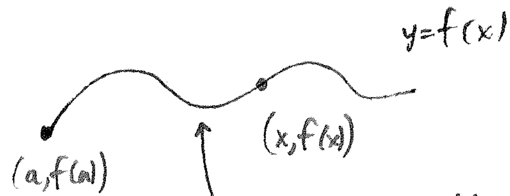
$$= \frac{1}{4}\sqrt{5}(2) + \frac{1}{4}\ln|\sqrt{5} + 2|$$

$$= \boxed{\frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}} \approx 1.4789$$

The arc length function

$$s(x) = s(f,x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

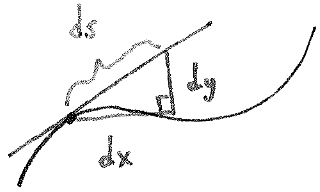
measures the length of a curve $y=f(x)$ starting at a.



By the fundamental theorem of calculus:

$$\frac{ds}{dx} = \frac{d}{dx}\int_a^x \sqrt{1+[f'(t)]^2}\, dt = \sqrt{1+[f'(x)]^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

so $\quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad$ or $\quad ds^2 = dy^2 + dx^2$



so ds is the change in the tangent direction of the curve. It's called a <u>length element</u>.

We can write: $L = \int ds$ to get a nice short form of the arc length formula.

Ex. let $f(x) = x^2 - \frac{1}{8}\ln(x)$ and $a=1$ the starting point.
Find the arc length function.

$$f'(x) = 2x - \frac{1}{8x}$$

$$f'(x)^2 = 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$1 + f'(x)^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1+f'(x)^2} = \sqrt{\left(2x+\frac{1}{8x}\right)^2} = 2x + \frac{1}{8x}$$

$$s(x) = \int_{a=1}^{x} 2t + \frac{1}{8t}\,dt = t^2 + \frac{1}{8}\ln t \Big|_{a=1}^{x} = \boxed{x^2 + \frac{1}{8}\ln x - 1}$$

Now, find the length of the curve from 1 to e

$$s(e) = e^2 + \frac{1}{8}\ln e - 1 = \boxed{e^2 - 1}$$

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373.$$