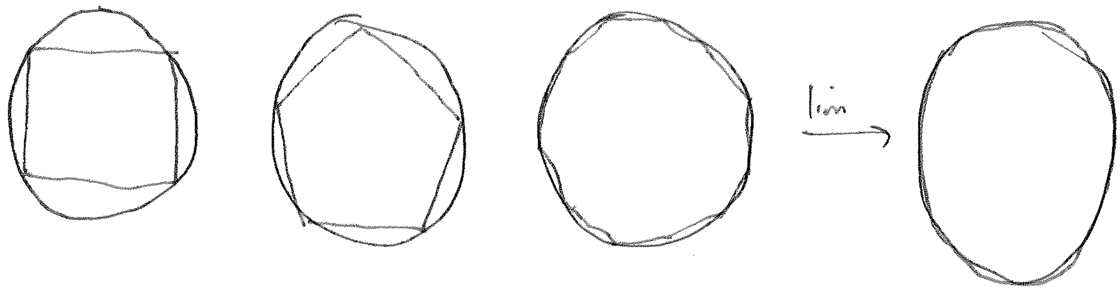
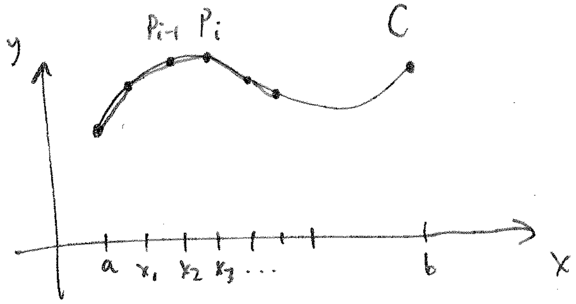
7.4. Arc Length

Idea. Circumference of a circle:



For a curve:



1. Partition $[a,b]$ just as you would for a Riemann sum.
2. "Connect the dots" with line segments.
3. The length of the segments added up is approximately the length of C.
4. Take a limit to make it exact.

$$|P_{i-1}P_i| = \sqrt{(x_{i-1}-x_i)^2 + (y_{i-1}-y_i)^2}$$
$$= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|}$$

For a uniform partition Δx_i is constant $= \Delta x$.

By the Mean Value Theorem (for derivatives), there is a x_i^* in $[x_{i-1}, x_i]$ such that
$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_{i-1} - x_i)$$

OR
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

so,
$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$
$$= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2}$$
$$= \Delta x\sqrt{1 + (f'(x_i^*))^2}$$

and
$$L = \lim_{n\to\infty} \sum_{n=1}^{\infty} \sqrt{1+f'(x_i^*)^2}\,\Delta x = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \boxed{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

Ex. Find the arc length of the curve $y^2 = x^3$ between $(1,1)$ and $(4,8)$.

$$y = x^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$$

$$L = \int_0^2 \sqrt{1 + \frac{9}{4}x}\, dx \qquad \begin{aligned} u &= 1 + 9/4\, x \\ du &= 9/4\, dx \end{aligned} \qquad \begin{aligned} u(0) &= 1 \\ u(2) &= \tfrac{22}{4} \end{aligned}$$

$$= \frac{4}{9}\int_1^{\frac{22}{4}} \sqrt{u}\, du = \frac{4}{9}\cdot\frac{2}{3}\, u^{3/2}\Big|_1^{22/4} = \frac{8}{27}\left(\left(\frac{22}{4}\right)^{3/2} - 1\right) \approx 3.5255$$

Exercise. Show that $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

~~if the length~~

if C is given by an invertible function between $P(a,c)$ and $Q(b,d)$.

Ex. Find the arc length of $x = y^2$ from $(0,0)$ to $(1,1)$.
or $y = \sqrt{x}$

$$\left.\begin{aligned} x &= y^2 \\ \frac{dx}{dy} &= 2y \\ \left(\frac{dx}{dy}\right)^2 &= 4y^2 \end{aligned}\right\} \quad L = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

let $y = \frac{1}{2}\tan\theta \Rightarrow \theta = \arctan(2y)$
$dy = \frac{1}{2}\sec^2\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\operatorname{atan}(2)} \sec^3\theta\, d\theta = \cancel{\frac{1}{2}\int_0^{\operatorname{atan}(2)} (1 + \tan^2\theta)\sec\theta\, d\theta}$$

$$\cancel{= \frac{1}{2}\int_0^{\operatorname{atan}(2)} \sec\theta\, d\theta + \frac{1}{2}\int_0^{\operatorname{atan}(2)} \tan^2\theta\sec\theta\, d\theta}$$