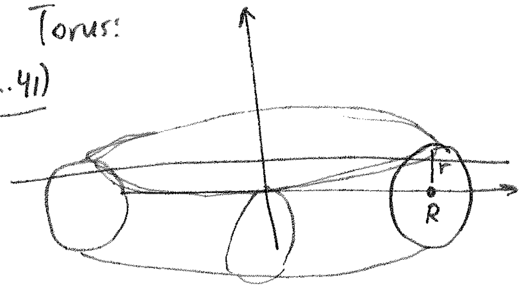
7.3. Volume by Cylindrical Shells

Torus:

7.2.41)



Horizontal slice

Equation of circle $(x-R)^2 + y^2 = r^2$

Integrate in y-direction: $(x-R)^2 = r^2 - y^2$

$$x - R = \pm\sqrt{r^2-y^2}$$

$$x = R \pm \sqrt{r^2-y^2}$$

outer radius: $R + \sqrt{r^2-y^2}$

inner radius: $R - \sqrt{r^2-y^2}$

Area: $\pi\left[\left(R+\sqrt{r^2-y^2}\right)^2 - \left(R-\sqrt{r^2-y^2}\right)^2\right]$

$$= \pi\left[\cancel{R^2} + 2R\sqrt{r^2-y^2} \cancel{+r^2} \cancel{-y^2} - \cancel{R^2} + 2R\sqrt{r^2-y^2} \cancel{-r^2} \cancel{+y^2}\right]$$

$$= 4\pi R\sqrt{r^2-y^2}$$

Volume: $4\pi R\int_{-r}^{r}\sqrt{r^2-y^2}\,dy = 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$

let $y = r\sin\theta$ $\quad\theta(0) = 0$

$dy = r\cos\theta\, d\theta \quad \theta(r) = \frac{\pi}{2}$

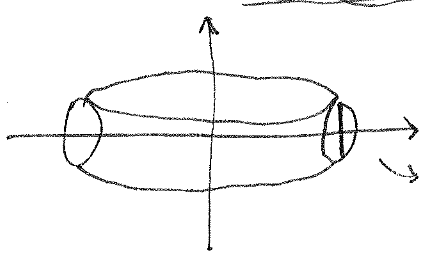
$$= 8\pi R\int_0^{\pi/2} r^2\cos^2\theta\,d\theta$$

$$= 8\pi R r^2 \cdot \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2\theta\,d\theta$$

$$= 4\pi R r^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \boxed{2\pi^2 R r^2}$$

Another way:



Take a vertical slice, and rotate it around. Get a cylindrical shell.

Area $= 2\pi r h$

$r = x$

$h = 2\sqrt{r^2-(x-R)^2}$

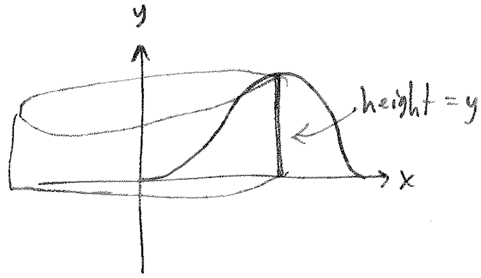
$$V = 4\pi\int_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\,dx$$

Ex. $y = 2x^2 - x^3$, $y = 0$ rotate around y-axis.

Find intersections: $2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x = 0 \quad x = 2$

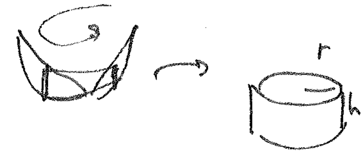


$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx = 2\pi \int_0^2 2x^3 - x^4\,dx$$

$$= 2\pi\left(\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right)\Big|_0^2 = 2\pi\left(\tfrac{16}{2} - \tfrac{32}{5}\right)$$

$$= 2\pi\left(\frac{80-64}{10}\right) = \boxed{\frac{16\pi}{5}}$$

Ex. $y = x$, $y = x^2$ rotate around y-axis.



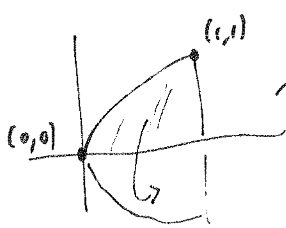
$r = x \quad h = x - x^2$

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 x^2 - x^3\,dx$$

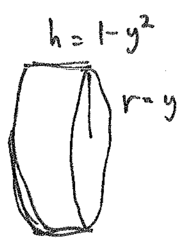
$$= 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1$$

$$= 2\pi\left(\frac{4-3}{12}\right) = \boxed{\frac{\pi}{6}}$$

Ex. $y = \sqrt{x}$ from $x = 0$ to 1 rotated around x-axis.



shell:

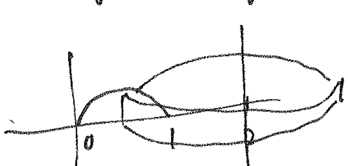


$$V = \int_0^1 2\pi y(1 - y^2)\,dy$$

$$= 2\pi \int_0^1 y - y^3\,dy$$

$$= 2\pi\left(\tfrac{1}{2}y^2 - \tfrac{1}{4}y^4\right)\Big|_0^1 = 2\pi\left(\tfrac{1}{4}\right) = \boxed{\pi/2}$$

Ex. $y = x - x^2$ $y = 0$ around line $x = 2$



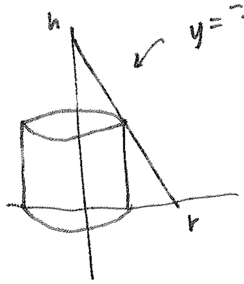
$$\left.\begin{array}{l} r = 2 - x \\ h = x - x^2 \end{array}\right\} \quad V = \int_0^1 2\pi(2-x)(x-x^2)\,dx = 2\pi \int_0^1 2x - x^2 - 2x^2 + x^3\,dx$$

$$= 2\pi\left[x^2 - x^3 + \tfrac{1}{4}x^4\right]_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

Ex. Right Circular Cone w/ height h and base radius r.

$\boxed{41}$



$$y = -\frac{h}{r}x + h$$

$$V = 2\pi \int_0^r x\left(-\frac{h}{r}x + h\right) dx$$

$$= 2\pi \int_0^r -\frac{h}{r}x^2 + hx \, dx$$

$$= 2\pi \left(\frac{-h}{3r}x^3 + \frac{h}{2}x^2\right)\Big|_0^r = 2\pi\left(-\frac{1}{3}hr^2 + \frac{1}{2}hr^2\right)$$

$$= \frac{2\pi}{6}hr^2 = \boxed{\frac{\pi}{3}hr^2}$$

Back to torus:

$$u = x - R \qquad u(R-r) = -r \qquad x = u + R$$
$$du = dx \qquad u(R+r) = +r$$

$$V = 4\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\, dx$$

$$= 4\pi \int_{-r}^{r} (u+R)\sqrt{r^2-u^2}\, du = 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi\int_{-r}^{r} R\sqrt{r^2-u^2}\,du$$

$$= \ldots$$

9/24/12