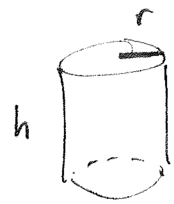
7.1 : Volumes

Consider a cylinder:

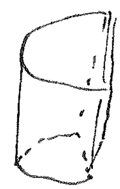


$V = \underbrace{\pi r^2}_{\text{Area of a cross section}} h$

Idea: Slice out a cross section, find its area, and integrate in the direction of the height.

$$V = \int_a^b \text{Area } d\,\text{height}$$

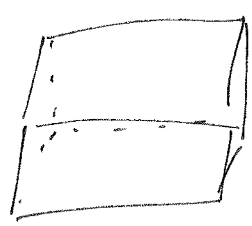
A more general cylinder

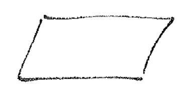


slice looks like

Find area, add 'em up.

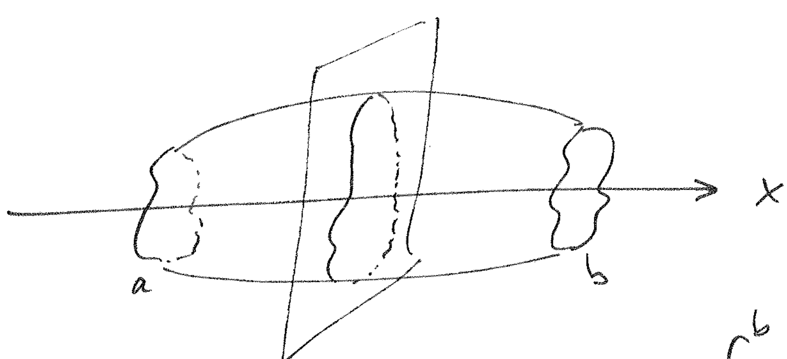
A "rectangular cylinder" (a box):



Slice 

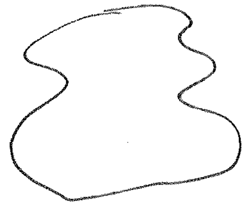
$$V = \int_a^b (l \cdot w)\,dh$$

You can imagine doing this for more exotic shapes:

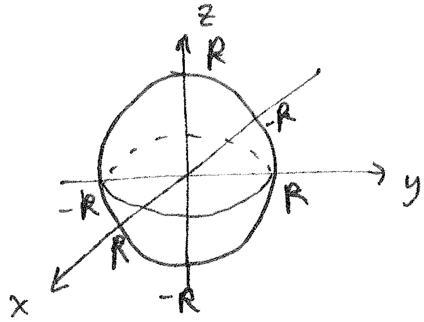


$$V = \int_a^b A(\text{slice})\,dx$$

Slice:
Area depends on
x

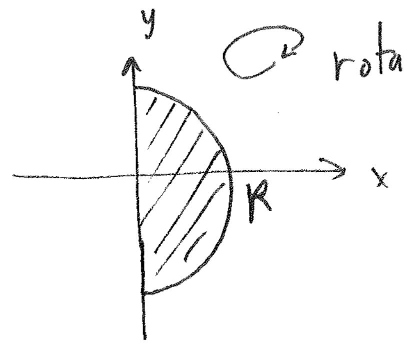


Ex. Volume of a sphere (ball) w/ radius R



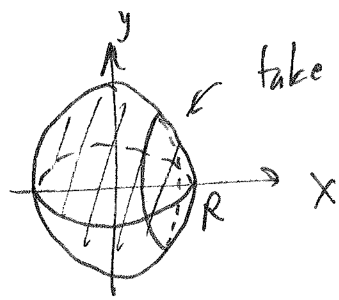
Not necessarily a good to think of it this way.

Instead think:

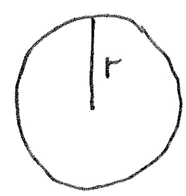


rotated around the y-axis 2π rads.

to become



take out a vertical slice:



Area $= \pi r^2$

Can we find $r = r(x)$?

Circle given by $x^2 + y^2 = R^2$

so $y = \sqrt{R^2 - x^2}$ in QI, and $r(x) = y = \sqrt{R^2 - x^2}$

So, area of slice $= \pi r^2 = \pi\left(\sqrt{R^2 - x^2}\right)^2 = \pi(R^2 - x^2)$

Integrate in x-direction from $-R$ to R:

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx = 2\pi \int_0^R R^2 - x^2\,dx$$

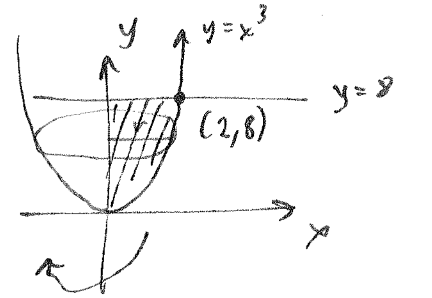
$$= 2\pi\left(R^2 x - \tfrac{1}{3}x^3\right)\Big|_0^R$$

$$= 2\pi\left(R^3 - \tfrac{1}{3}R^3\right) = 2\pi\left(\tfrac{2}{3}R^3\right)$$

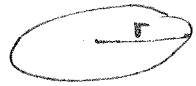
$$= \boxed{\tfrac{4}{3}\pi R^3} \; ! \; :)$$

Ex. $y = x^3$, $y = 8$, $x = 0$

Revolve around y-axis.



Slice horizontally:

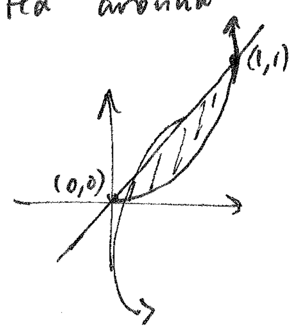


$$r(y) = \sqrt[3]{y}$$

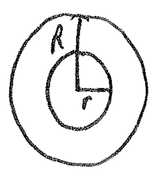
$$V = \int_0^8 \pi r^2\, dy = \pi \int_0^8 y^{2/3}\, dy = \frac{3\pi}{5} y^{5/3} \Big|_0^8 = \frac{3\pi}{5}(32)$$

$$= \boxed{\frac{96\pi}{5}}$$

Ex. R is enclosed by $y = x$ and $y = x^2$, then rotated around x-axis.



Vertical slice:



$$A = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

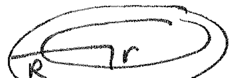
$$\left.\begin{array}{l} R(x) = x \\ r(x) = x^2 \end{array}\right\} A(x) = \pi (x^2 - x^4)$$

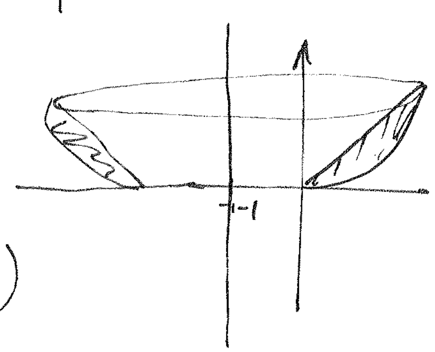
$$\cancel{\text{Then } V = \pi \int_0^1 x - x^2\, dx = \pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}}$$

$$V = \pi \int_0^1 x^2 - x^4\, dx = \pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{2\pi}{15}}$$

Ex. Rotate the same R about $x = -1$

Horizontal slice:





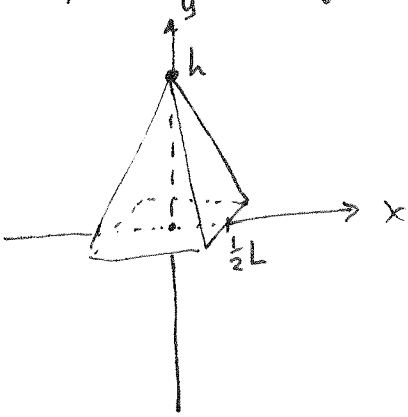
$$\left.\begin{array}{l} R(y) = \sqrt{y} + 1 \\ r(y) = y + 1 \end{array}\right\} A(y) = \pi \left((\sqrt{y}+1)^2 - (y+1)^2\right)$$

$$= \pi (y + 2\sqrt{y} \cancel{+1} - y^2 - 2y \cancel{-1})$$

$$= \pi (-y^2 - y + 2\sqrt{y})$$

$$V = \pi \int_0^1 -y^2 - y + 2y^{1/2}\, dy = \pi \left(-\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{4}{3}y^{3/2}\right)\Big|_0^1 = \pi \left(\frac{4}{3} - \frac{1}{3} - \frac{1}{2}\right) = \boxed{\frac{\pi}{2}}$$

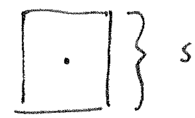
Ex. Pyramid w/ square base, side length of base $= L$, and height $= h$.

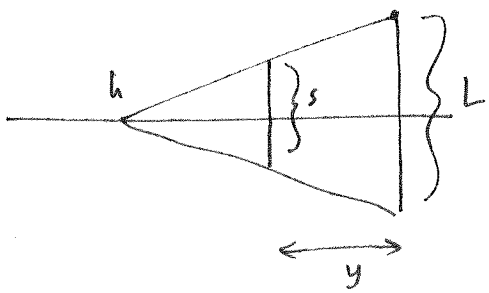


need eqn of line w/ y-int h, x-int $\frac{1}{2}L$.

$$y = -\frac{2h}{L}x + h \Rightarrow x = \frac{-L}{2h}y + \frac{L}{2}$$

Horizontal slice:





(turn it around)

By similar triangles:

$$s = 2x = \frac{-L}{h}y + L$$

Area of square is s^2

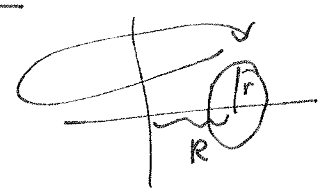
$$s^2 = \left(-\frac{L}{h}y + L\right)^2 = \frac{L^2}{h^2}y^2 - \frac{2L^2}{h}y + L^2$$

Then $$V = \int_0^h \frac{L^2}{h^2}y^2 - \frac{2L^2}{h}y + L^2\,dy = \left(\frac{L^2}{3h^2}y^3 - \frac{L^2}{h}y^2 + L^2 y\right)\Big|_0^h$$

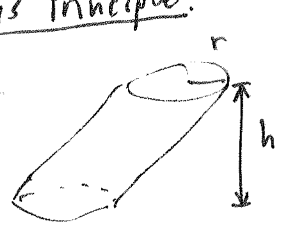
$$= \frac{L^2}{3h^2}h^3 - \frac{L^2}{h}h^2 + L^2 h$$

$$= \frac{1}{3}L^2h - L^2h + L^2h = \boxed{\frac{1}{3}L^2h} \quad ! \quad \ddot{\smile}$$

HW. Volume of a torus.



Cavalieri's Principle:



$V = \pi r^2 h$