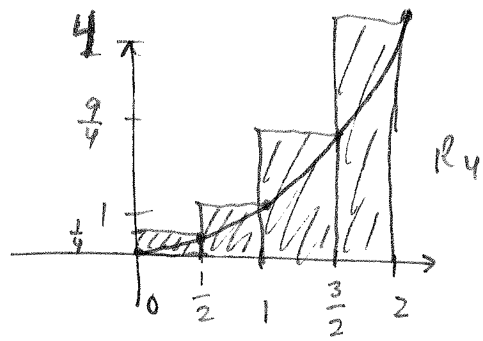
6.5 Approximate Integration

Recall: Riemann Sum definition of an integral.

- Calc 1 integration notes added to web page. Worth looking at if you need a Riem. sum refresher. Figures missing. Draw them yourself or ask me what they are supposed to look like.

Ex. $\int_0^2 x^2\,dx$



$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$

$$R_4 = \sum_{j=1}^{4} f(x_j)\,\Delta x_j$$

$\Delta x_j = \begin{cases} \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} \\ x_i = a + i\Delta x \end{cases}$

$$= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{4}\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2} + \tfrac{9}{4}\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{8} + \tfrac{1}{2} + \tfrac{9}{8} + 2 = \frac{1+4+9+16}{8} = \boxed{\frac{30}{8}}$$ over estimate

$\left.\begin{aligned} L_4 &= \\ M_4 &= \end{aligned}\right\}$ review.

Trapezoidal Rule: $\boxed{\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right] \text{ where } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x}$

Trap for: $\int_0^2 x^2\,dx$

Error: Neither midpoint nor trap rule give the exact answer. The exact error is given by:

$$E_{M_n} = \int_a^b f(x)\,dx - M_n\ , \quad E_{T_n} = \int_a^b f(x)\,dx - T_n$$

Error bounds: Suppose $f''(x) \leq K$ for all $a \leq x \leq b$. Then

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_{M_n}| \leq \frac{K(b-a)^3}{24n^2}.$$

In particular, this tells us how many n we need to use to guarantee a certain error level.

Ex. We want to use T_n to estimate $\int_0^2 x^2\,dx$ with error at most $\frac{1}{100}$.

$$f''(x) = 2 = K, \quad \text{so} \quad |E_{T_n}| \leq \frac{2(b-a)^3}{12n^2} \leq \frac{1}{100}$$

$$\frac{1}{100} \geq \frac{2(2^3)}{12n^2} = \frac{16}{12n^2}$$

$$\Rightarrow n^2 \geq \frac{1600}{12} = \frac{400}{3}$$

$$\text{so } n \geq \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}} \approx 11.5$$

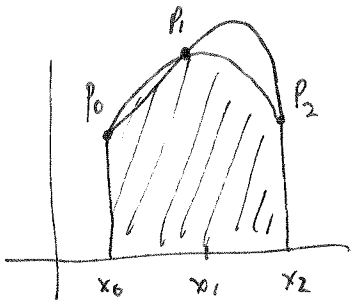
So we must use at least 12 boxes to get this accuracy. Since we're lazy, we should use exactly 12.

On computer: $\int_0^1 e^{-x^2}\,dx$

#5 $\int_0^{\pi} x^2 \sin x\, dx$ $n=8$ M, T, S

9/12/12

Simpson's Rule: A combination of T_n and M_n (Ex. 40).



Uses a parabola that passes thru P_0, P_1, and P_2.

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

where n is even and $\Delta x = \frac{b-a}{n}$

Exs above.

Proof / Derivation in book pp. 338-9.

Error bound for S_n:

Suppose $|f^{(4)}(x)| \le K$ for $a \le x \le b$. Then

$$|E_{S_n}| \le \frac{K(b-a)^5}{180\, n^4}$$

Ex (20). How large must n be to guarantee that Simpsons rule approximates $\int_0^1 e^{x^2}\,dx$ to within 0.00001?