Math 243: Calculus II
Exam 1: Ch 10, pt 1

Due Date: Friday, 31 August 2012

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded not only on whether the answer is correct, but also if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots in terms of such (no rounded off decimals).

1. Complete a few squares to show that the equation represents a sphere, and find the center and radius.

$$x^2 + y^2 + z^2 = 4x - 2y + 6z - 5$$

$x^2 - 4x + 4 \quad + y^2 + 2y + 1 \quad + z^2 - 6z + 9 = -5 + 4 + 1 + 9$

$\boxed{(x-2)^2 + (y+1)^2 + (z-3)^2 = 9}$

center: $C(2, -1, 3)$

radius: $r = \sqrt{9} = 3$

2. Find an equation of a sphere if one of its diameters has endpoints $P(2, 1, 4)$ and $Q(4, 3, 10)$

Center = midpt of $\overline{PQ}$ = $C(3, 2, 7)$

$r = d(C, P) = \sqrt{1 + 1 + 9} = \sqrt{11}$

eqn: $\boxed{(x-3)^2 + (y-2)^2 + (z-7)^2 = 11}$

3. Find a unit vector that has the same direction as $\boldsymbol{v} = \langle 2, -4, 4\rangle$.

$$|\vec{v}| = \sqrt{4+16+16} = \sqrt{36} = 6$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{6}\langle 2, -4, 4\rangle = \left\langle \frac{2}{6}, \frac{-4}{6}, \frac{4}{6}\right\rangle = \boxed{\left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right\rangle}$$

4. If $\boldsymbol{v} \in \mathbb{V}^2$ is in quadrant IV, makes an angle of $\frac{\pi}{6}$ with the positive x-axis, and has a length of 2, write $\boldsymbol{v}$ in its component form.

$$\vec{v} = r\langle \cos\theta, -\sin\theta\rangle = 2\left\langle \cos\left(\frac{\pi}{6}\right), -\sin\left(\frac{\pi}{6}\right)\right\rangle$$

$$= 2\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$$

$$= \boxed{\langle \sqrt{3}, -1\rangle}$$

5. Let $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$. Prove that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= b_1 a_1 + b_2 a_2 + b_3 a_3$$

$$= \vec{b} \cdot \vec{a}$$

6. Find $\boldsymbol{a} \cdot \boldsymbol{b}$ for $\boldsymbol{a} = 4\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$.

$\vec{a} = \langle 0, 4, -3 \rangle$

$\vec{b} = \langle 2, 4, 6 \rangle$

$$\vec{a} \cdot \vec{b} = 2 \cdot 0 + 4 \cdot 4 + (-3) \cdot 6$$
$$= 0 + 16 - 18$$
$$= \boxed{-2}$$

7. Find a vector that is orthogonal to both $\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{k}$.

$\vec{a} = \langle 0, 1, 1 \rangle$

$\vec{b} = \langle 1, 0, 1 \rangle$

$$\boxed{\vec{a} \times \vec{b} = \langle 1, 1, -1 \rangle}$$

8. Find $\text{proj}_{\boldsymbol{a}} \boldsymbol{b}$ for $\boldsymbol{a} = \langle 3, -6, 2 \rangle$ and $\boldsymbol{b} = \langle 1, 2, 3 \rangle$.

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{-3}{49} \langle 3, -6, 2 \rangle = \boxed{\left\langle \frac{-9}{49}, \frac{18}{49}, \frac{-6}{49} \right\rangle}$$

$\vec{a} \cdot \vec{b} = 3 - 12 + 6 = -3$

$\vec{a} \cdot \vec{a} = 9 + 36 + 4 = 49$

9. Prove the Cauchy-Schwarz Inequality: $|\boldsymbol{a} \cdot \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$. Include all necessary details.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$0 \leq |\cos\theta| \leq 1$

so, $|\vec{a} \cdot \vec{b}| = \left| |\vec{a}||\vec{b}| \cos\theta \right|$

$= |\vec{a}||\vec{b}| |\cos\theta|$

$\leq |\vec{a}||\vec{b}|$ $\square$

10. Find $\boldsymbol{a} \times \boldsymbol{b}$ for $\boldsymbol{a} = \langle 5, 1, 4 \rangle$ and $\boldsymbol{b} = \langle -1, 0, 2 \rangle$. Then show that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

$\vec{a} \times \vec{b} = \langle 2, -14, 1 \rangle$

$(\vec{a} \times \vec{b}) \cdot \vec{a} = 10 - 14 + 4 = 0$