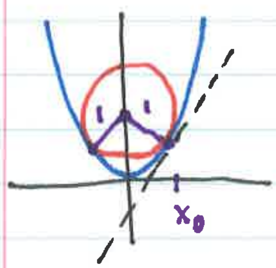
Calc I Ch 2 XC

9. 

The point (x_0, x_0^2) lies on both curves, and on their shared tangent line.

Compute the derivatives:

parabola: $\frac{dy}{dx} = 2x$

circle: $\frac{d}{dx}\left[x^2 + (y-k)^2 = 1\right]$

$$2x + 2(y-k)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y-k}$$

Set these equal to obtain $2x = \frac{-x}{y-k} \Rightarrow y-k = -\frac{1}{2}$

Then plug into the equation of the circle to get

$$x^2 + \left(-\tfrac{1}{2}\right)^2 = 1 \quad \Rightarrow \quad x_0 = \pm\frac{\sqrt{3}}{2}.$$

Then $y_0 = x_0^2 = \frac{3}{4}$.

The center of the circle is the y-intercept of the normal line through this point:

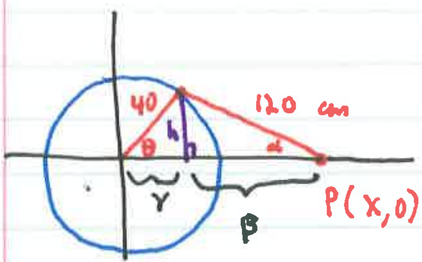
$$y = y_0 - \frac{1}{dy/dx}(x - x_0)$$

$$y = \frac{3}{4} - \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right)$$

$$y(0) = \frac{3}{4} - \frac{1}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}.$$

Thus, the center of the circle is at $(0, 5/4)$.

13.



$$\frac{d\theta}{dt} = 360 \frac{\text{rot}}{\text{min}} = 720\pi \frac{\text{rad}}{\text{min}} = 43200\pi \frac{\text{rad}}{\text{sec}}$$

On one hand, $\sin\theta = \frac{h}{40} \Rightarrow h = 40 \sin\theta$

On the other, $\sin\alpha = \frac{h}{120} \Rightarrow h = 120 \sin\alpha$

Therefore, $\sin\alpha = \frac{1}{3}\sin\theta$.

Take d/dt implicitly,

$$\frac{d}{dt}\left[\sin\alpha = \frac{1}{3}\sin\theta\right]$$

$$\cos\alpha \, d\alpha/dt = \frac{1}{3}\cos\theta \, d\theta/dt$$

$$\frac{d\alpha}{dt} = \frac{\cos\theta}{3\cos\alpha}\frac{d\theta}{dt}$$

Now, $\cos\alpha = \beta/120$ where $\beta = \sqrt{120^2 - h^2} = \sqrt{120^2 - 40^2\sin^2\theta}$

$$= 40\sqrt{9-\sin^2\theta}$$

Therefore, $\frac{d\alpha}{dt} = \frac{\cos\theta}{120\sqrt{9-\sin^2\theta}}\frac{d\theta}{dt}$

when $\theta = \pi/3$,

$$\frac{d\alpha}{dt} = \frac{\cos(\pi/3)}{120\sqrt{9-\sin^2(\pi/3)}} \cdot 43200\pi = \frac{\frac{1}{2}\cdot 43200\pi}{120\sqrt{9-3/4}}$$

$$= \frac{180\pi}{\sqrt{33/4}} = \boxed{\frac{360\pi}{\sqrt{33}} \frac{\text{rad}}{\text{sec}} = \frac{d\alpha}{dt}}$$ a.)

Next, $x = |OP| = r + \beta = 40\cos\theta + 40\sqrt{9-\sin^2\theta} = \boxed{40\left(\cos\theta + \sqrt{9-\sin^2\theta}\right) = x}$ b.)

The velocity of x is $\frac{dx}{dt} = 40\left(-\sin\theta\frac{d\theta}{dt} - \frac{2\sin\theta\cos\theta}{\sqrt{9-\sin^2\theta}}\frac{d\theta}{dt}\right)$

$$\boxed{\frac{dx}{dt} = -172800\pi\sin\theta\left(1 + \frac{2\cos\theta}{\sqrt{9-\sin^2\theta}}\right)}$$ c.)

15. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \rightarrow \quad y^2 = 4\left(1 - \frac{x^2}{9}\right)$, in QI : $y = 2\sqrt{1 - x^2/9}$.

$\frac{d}{dx}: \left[\frac{2}{9}x + \frac{1}{2}y\frac{dy}{dx} = 0 \right.$ $\qquad$ or $y = \frac{2}{3}\sqrt{9-x^2}$

$$\frac{dy}{dx} = \frac{-4x}{9y} = \frac{-4x}{9(2/3)\sqrt{9-x^2}} = \frac{-2x}{3\sqrt{9-x^2}}$$

The equation of the tangent line is:

$$y = \frac{2}{3}\sqrt{9-x_0^2} + \frac{-2x_0}{3\sqrt{9-x_0^2}}(x - x_0)$$

x_T is the solution of the equation:

$$\cancel{\frac{2}{3}}\sqrt{9-x_0^2} = \cancel{\frac{2}{3}}\frac{x_0}{\sqrt{9-x_0^2}}(x - x_0)$$

$$\frac{9-x_0^2}{x_0} = x - x_0$$

$$x = \frac{9-x_0^2}{x_0} + x_0 \text{ , so } x_T = 9/x_0$$

$$\boxed{\lim_{x_0 \to 0^+} x_T = \lim_{x_0 \to 0^+} \frac{9}{x_0} = +\infty , \quad \lim_{x_0 \to 3^-} x_T = \lim_{x_0 \to 3^-} \frac{9}{x_0} = \frac{9}{3} = 3.}$$

x_N is the solution of the equation:

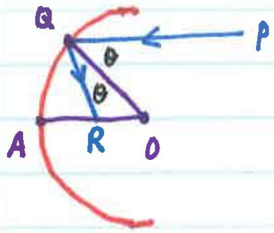
$$\frac{2}{3}\cancel{\sqrt{9-x_0^2}} = \frac{-3}{2}\frac{\cancel{\sqrt{9-x_0^2}}}{x_0}(x - x_0)$$

$$-4/9\, x_0 = x - x_0$$

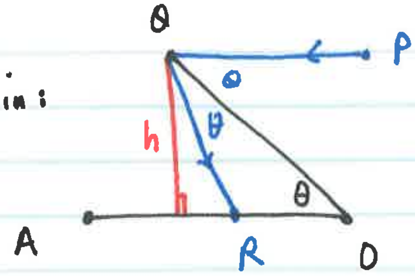
so, $x_N = \frac{5}{9}x_0$

$$\boxed{\lim_{x_0 \to 0^+} x_N = \lim_{x_0 \to 0^+} \frac{5}{9}x_0 = 0, \quad \lim_{x_0 \to 3^-} x_N = \lim_{x_0 \to 3^-} \frac{5}{9}x_0 = \frac{5}{9} \cdot 3 = \frac{5}{3}}$$

19.

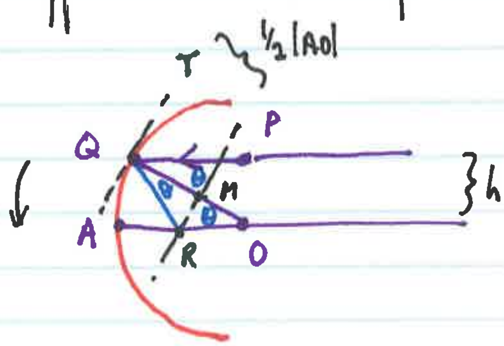


Zoom in:



The problem wants us to compute $\lim_{h \to 0^+} R$.

Since ΔOQR is isosceles, then R lies on the perpendicular bisector of $\overline{OQ}$. Since $\overline{OQ}$ is a radius of the circle, it is perpendicular to the tangent line to the circle at Q. If M is the midpoint of $\overline{OQ}$, then $\overline{MR}$ is parallel to the tangent line at Q, and $\frac{1}{2}|OQ|$ units away.
As $h \to 0$, $Q \to A$, and the tangent line becomes more vertical.
Therefore R approaches the midpoint of $\overline{AO}$.



To show this using calculus, turn the picture around and use the unit circle.

$y = \sqrt{1-x^2}$, $A = (0,1)$, $Q = (x_0, \sqrt{1-x_0^2})$, $M = (x_0, \sqrt{1/4 - x_0^2})$
and $R = (0,k)$ is the y-intercept of the line

$$y = \sqrt{1/4 - x_0^2} + \frac{-x_0}{\sqrt{1-x_0^2}}(x - x_0), \text{ so}$$

$$k = \sqrt{1/4 - x_0^2} + \frac{x_0^2}{\sqrt{1-x_0^2}}$$

$$\lim_{x_0 \to 0^+} k = \lim_{x_0 \to 0^+} \sqrt{1/4 - x_0^2} + \frac{x_0^2}{\sqrt{1-x_0^2}} = \sqrt{1/4} + \frac{0}{1} = \frac{1}{2}.$$

Therefore $\lim_{h \to 0^+} R = (0, 1/2)$.

28. a.) Graph made w/ Sage.

b.) $f(x) = (x-a)(x-b)(x-c)$

$f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$

$$f'\left(\frac{a+b}{2}\right) = \left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{b-a}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{a-b}{2}\right)\left(\frac{b-a}{2}\right)$$

$$f\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right)$$

The tangent line at $x = \frac{a+b}{2}$ is then given by

$$y = \left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left[\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{b-a}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{a-b}{2}\right)\left(\frac{b-a}{2}\right)\right]\left(x - \frac{a+b}{2}\right)$$

and has root:

$$x = \left(\frac{a+b}{2}\right) - \frac{\left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right)}{\left[\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{b-a}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{a-b}{2}\right)\left(\frac{b-a}{2}\right)\right]}$$

And simplifying this is where Sage comes back into play.

(All of this calculation <u>can</u> be done in Sage, but it's good practice to do it by hand.)