Chapter 4 : Complex Numbers

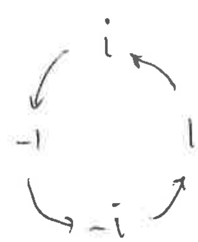
let $a, b \in \mathbb{R}$. The number $a+bi$ is called a complex number.

where $i = \sqrt{-1}$ by definition.

a is called the real part.

b is called the imaginary part.

Circle of i's:



Ex. $i^7 = -i$

$i^{-3} = i$

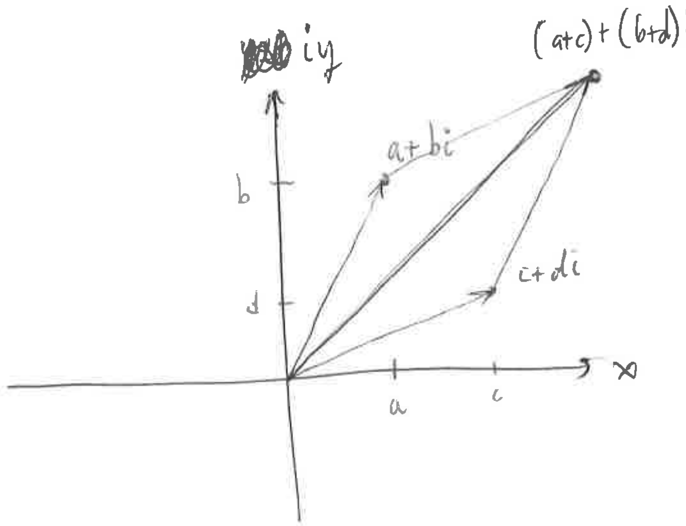
$i^{47} = (i^{44})i^{-3} = i^{-3} = -i$

$i^{-1} = \frac{1}{i} = -i$

Equality. $a + bi = c + di$ iff $a = c$ and $b = d$.

Geometrically:

Ex. plot a bunch of numbers.



In general, a complex number can be thought of as a pt in the plane. (Sound familiar?)

– Add complex numbers as though they were vectors.

Subtraction as well.

Formulas for $(a+bi)+(c+di)$ and

$(a+bi)-(c+di)$ (just distribute)

Ex. $(3+2i)+(4-i)-(7+i)$

Ex. $2i+(-3-4i)-(-3-3i)$

Complex numbers have more structure than just being vectors! In particular, we can multiply two of them to get back another vector. This makes $\mathbb{C}$ an algebra, rather than just a vector space. [$\mathbb{C}$ can also be thought of as a field in its own right.]

$$(a+bi)(c+di) = ac + adi + bci + bdi^2$$
$$= \underbrace{(ac-bd)}_{x} + \underbrace{(ad+bc)}_{y}i$$

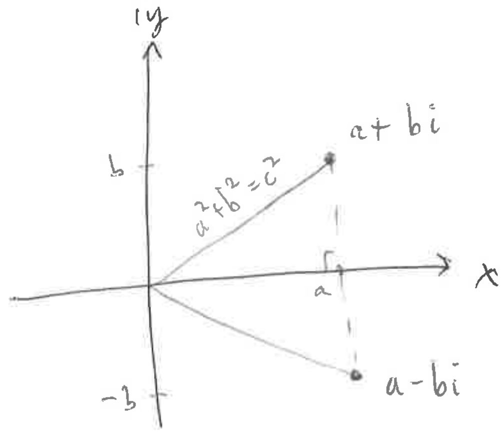
Ex. $(2-4i)(3+3i)$

$(4+5i)(4-5i)$

$(4+2i)^2$

Complex Conjugates

Geometrically:



$\overline{(a+bi)} = a-bi$: just reflect over the x ($\mathbb{R}$) axis.

So what? $(a+bi)\overline{(a+bi)} = (a+bi)(a-bi) = a^2 - (bi)^2$

$$= a^2 - b^2(-1)$$

$$= a^2 + b^2 \in \mathbb{R}.$$

A number times its conjugate is always real.

Ex. If $a \in \mathbb{R}$, then $\bar{a} = a$, and vice versa.

$1+i$

$4-3i$

Ex. Absolute value: geometrically: distance from the origin. i.e., $\sqrt{c^2} = \sqrt{(a+bi)(a-bi)} =: |a+bi|$

Thus, $|a+bi| = |a-bi|$, which should be the case.

Dividing Complex Numbers:

Idea: "rationalize" the denominator.

Ex. $$\frac{(2+3i)}{(4-2i)}\cdot\frac{(4+2i)}{(4+2i)} = \frac{(8-6)+(12+4)i}{16+4} = \frac{2+16i}{20} = \frac{1}{10} + \frac{4}{5}i$$

Ex. $$\frac{(2+i)}{(2-i)}\cdot\frac{(2+i)}{(2+i)} = \frac{(4-1)+(2+2)i}{2^2+1^2} = \frac{3+4i}{5} = \frac{3}{5} + \frac{4}{5}i$$

Solving eqns.

$$3x^2 - 2x + 5 = 0$$

$$x^3 + 1 = 0$$

Ex. $\sqrt{-6}\,\sqrt{-6} = ?$

Proofs. $$\overline{(a+bi)(c+di)} = \overline{(a+bi)} \cdot \overline{(c+di)}$$

$$\overline{(a+bi)+(c+di)} = \overline{(a+bi)} + \overline{(c+di)}.$$